Math 108, Fall 2013.
Nov. 22, 2013.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK AND FULLY JUSTIFY YOUR ANSWERS TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

1	
2	
3	
4	
5	
TOTAL	

1. For each of the following sets A and relations R on A, answer the following questions.

(a) $A = \{1, 2, 3, 4\}$, $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1), (3,3), (4,4)\}$. ~~You do not need to justify Yes answers, but you do have to justify No answers.~~

Is R reflexive? Yes.

Is R symmetric? Yes.

Is R transitive? No. $(3,1) \in R$ and $(1,2) \in R$, but $(3,2) \notin R$.

(b) $A = \mathbb{N} = \{1, 2, 3, \ldots\}$, xRy if and only if $x + 1 \leq y$. ~~Justify all answers.~~ 2

Is R reflexive? No. $(1,1) \notin R$, as $1+1 > 1$.

Is R symmetric? No. $(1,2) \in R$ (as $1+1 \leq 2$), but $(2,1) \notin R$ (as $2+1 > 1$).

Is R transitive? Yes. If $x+1 \leq y$ and $y+1 \leq z$, then $x+1 \leq z-1 \leq z$.

(c) $A = \{1, 2, 3, 4\}$, $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$. ~~You do not need to justify Yes answers, but you do have to justify No answers.~~

Is R reflexive? Yes.

Is R symmetric? Yes. 6

Is R transitive? Yes.

2. For each of the following sets A and relations R on A, first answer the question: is R a function from A to A? If the answer is yes, also determine whether the function is one-to-one and whether it is onto.

(a) $A = \{1, 2, 3, 4\}$, $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1), (3,3), (4,4)\}$.

No. $(1,1) \in R$ and $(1,2) \in R$, so no single-value property.

(b) $A = \mathbb{N} = \{1, 2, 3, \ldots\}$, xRy if and only if $(x-2)^2 = (y-2)^2$.

No. $(1,1) \in R$ and $(1,3) \in R$, so no single-value property.

(c) $A = \{1, 2, 3, 4\}$, $R = \{(1,2), (2,1), (3,3), (4,4)\}$.

Yes. The function is one-to-one and onto.

(d) $A = \{1, 2, 3, 4\}$, $R = \{(1,2), (2,2), (3,3), (4,4)\}$.

Yes. The function is neither one-to-one ($1 \to 2$ and $2 \to 2$), nor onto (1 is not in the range).

3. Define the relation R on $\mathbb{R}\times\mathbb{R}$ as follows: $(x,y)R(a,b)$ if and only if $y-x^2=b-a^2$. Prove that this is an equivalence relation and describe the resulting partition. In particular, sketch the equivalence classes of $(0,0)$, $(1,0)$, and $(1,2)$. (To show that R is an equivalence relation, you may use any method we covered in the lecture.)

Let $f(x,y) = y-x^2$, $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$.

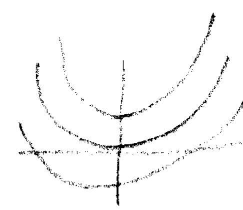
$(x,y)\ R\ (a,b)$ if and only if $f(x,y) = f(a,b)$. 10

By what we did in class, R is an equivalence relation.

$(0,0)|R = \{(x,y) : y-x^2=0\} = \{(x,y) : y=x^2\}$ 5

$(1,0)|R = \{(x,y) : y=x^2-1\}$

$(1,2)|R = \{(x,y) : y=x^2+1\}$



The equivalence classes partition the plane $\mathbb{R}\times\mathbb{R}$ into parabolas, which are translations (vertical) of $y=x^2$. 5

4. Assume A, B, and C are arbitrary sets and $f : A \to B$ and $g : B \to C$ are arbitrary functions. For each statement below, prove it or provide a counterexample.

(a) Prove: If $g \circ f$ is onto, then g is onto.

Assume $z \in C$. We need to find a $y \in B$ so that $g(y) = z$. 7

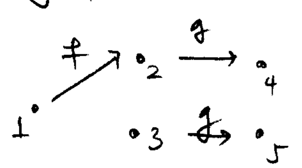
As $g \circ f$ is onto, there exists an $x \in A$, so that $g \circ f(x) = z$, i.e. $g(f(x)) = z$. Take $y = f(x)$.

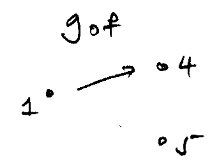
Then $y \in B$ and $g(y) = z$. $\square$

(b) Give a counterexample: Converse of (a).

Converse: g onto $\Rightarrow$ $g \circ f$ onto.

We need to find an example where g is onto, but $g \circ f$ is not onto. 6





g is onto $\qquad$ $g \circ f$ is not onto

(c) Prove: If f and g are both one-to-one, then $g \circ f$ is one-to-one.

Assume $(g \circ f)(x_1) = (g \circ f)(x_2)$ for some $x_1, x_2 \in A$.

Then $g(f(x_1)) = g(f(x_2))$. As g is one-to-one, 7

$f(x_1) = f(x_2)$ and then, as f is one-to-one,

$x_1 = x_2$ $\square$

5. Assume A and B are arbitrary sets, that $C \subseteq A$, $D \subseteq A$, $E \subseteq B$, $F \subseteq B$ are arbitrary subsets, and that $f : A \to B$ is an arbitrary function. For each statement below, prove it or provide a counterexample.

(a) Prove: If $C \subseteq D$, then $f(C) \subseteq f(D)$.

Take $y \in f(C)$. We need to prove $y \in f(D)$.
There exists an $x \in C$ so that $y \in f(x)$. But
then $x \in D$, and so $y \in f(D)$.

(b) Prove: $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.

$$\begin{aligned} x \in f^{-1}(E \cap F) &\iff f(x) \in E \cap F \\ &\iff (f(x) \in E) \wedge (f(x) \in F) \\ &\iff (x \in f^{-1}(E)) \wedge (x \in f^{-1}(F)) \\ &\iff x \in f^{-1}(E) \cap f^{-1}(F) \end{aligned}$$

(c) Give a counterexample: If $E \neq \emptyset$, then $f^{-1}(E) \neq \emptyset$.

$$1 \bullet \xrightarrow{f} \bullet 2 \qquad \bullet 3$$

$E = \{3\} \neq \emptyset$

$f^{-1}(E) = \emptyset$

(d) Give a counterexample: $f^{-1}(f(C)) \subseteq C$.

$$\begin{tikzcd} 1 \bullet \arrow[r, "f"] & \bullet 3 \\ 2 \bullet \arrow[ur] & \end{tikzcd}$$

$C = \{1\}$, $f(C) = \{3\}$

$f^{-1}(f(C)) = f^{-1}(\{3\}) = \{1, 2\}$

$\neq C$