Lecture 1:

Shapes in Deterministic and Random Rules

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Local Monotone Growth CA.

The occupied set $A_t \subset \mathbf{Z}^2$ evolves in discrete time $t = 0, 1, \ldots$, with A_0 some (usually large enough *finite*) set. The rule is determined by

- a finite *neighborhood* $\mathcal{N}$, with $0 \in \mathcal{N}$,
- a set of probabilities $\pi(S) \in [0, 1]$, $S \subset \mathcal{N}$.

Then, given A_t,

> the updated set A_{t+1} of occupied points is obtained by adjoining, independently, every $x \in \mathbf{Z}^d$ with probability $\pi((A_t - x) \cap \mathcal{N})$.

Common assumptions:

- $\pi(\emptyset) = 0$,
- π is *symmetric*: $-\mathcal{N} = \mathcal{N}$ and $\pi(-S) = \pi(S)$,
- π *solidifies*: $0 \in S \implies \pi(S) = 1$ (i.e., $A_t \subset A_{t+1}$),
- π is *monotone* (*attractive*): $S_1 \subset S_2 \implies \pi(S_1) \leq \pi(S_2)$.

Threshold Growth Model (TGM).

This is a *totalistic* monotone growth CA. There exist a

- a *threshold* $\theta \geq 1$, and
- update probabilities $0 < p_\theta \leq p_{\theta+1} \leq \cdots \leq p_{|\mathcal{N}|-1}$,

so that

(1) $A_t \subset A_{t+1}$,

(2) $x \notin A_t$ belongs to A_{t+1} with probability $p_{|A_t \cap (x+\mathcal{N})|}$.

Often, $p_i \equiv p$.

For example, $\mathcal{N}$ could consist of the 4 nearest sites to the origin (*von Neumann* neighborhood) or 8 nearest sites (*Moore* neighborhood). Large neighborhoods $\mathcal{N} = \mathcal{N}_\rho$ will often be range ρ *box* neighborhoods, $(2\rho+1)\times(2\rho+1)$ boxes centered at 0. More generally, $\mathcal{N}_\rho = \{x : ||x|| \leq \rho\}$, where $||\cdot||$ is some norm.

Deterministic dynamics.

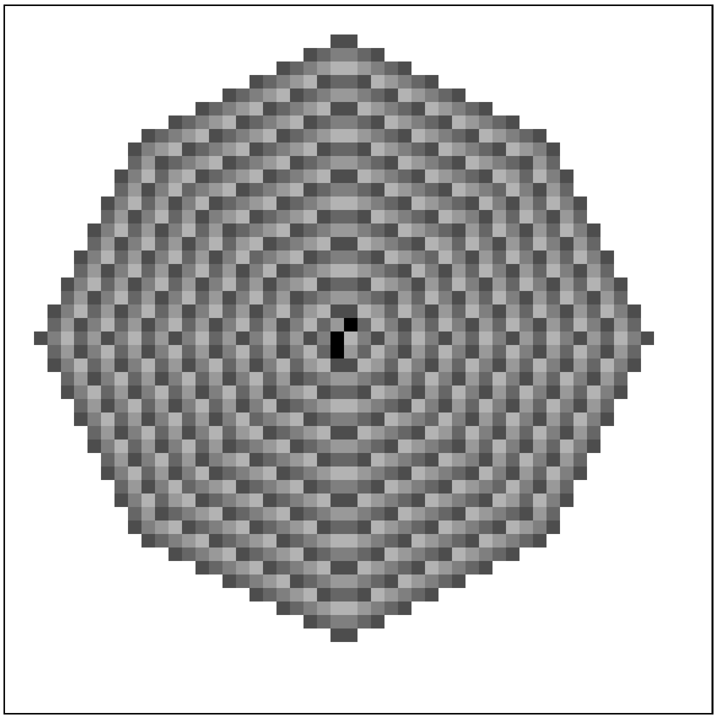
Take a range ρ box TGM with $p_\theta = 1$. There are only 2 possibilities (Bohman, 1999):

- either A_t stop growing: $A_{t+1} = A_t$ for some t,
- or $A_\infty = \mathbf{Z}^2$ and A_t/t converges, as $t \to \infty$, to the limiting shape $L = L_1$.

The convergence to limiting set L_1 is very fast. In fact, there exists a constant $C = C(\mathcal{N}, A_0)$ so that A_t differs from tL by C (Willson, 1978). This C is, for appropriate initial sets, of order ρ.

In general, the above is not true, e.g., $\theta = 3$ and

$$x + \mathcal{N} = \begin{array}{ccccccc} & & & \bullet & & & \\ & & & \bullet & & & \\ & & & \bullet & & & \\ \bullet & \bullet & \bullet & x & \bullet & \bullet & \bullet \\ & & & \bullet & & & \\ & & & \bullet & & & \\ & & & \bullet & & & \end{array}, \qquad A_0 = \begin{array}{ccc} & \square & \\ \square & \square & \square \\ & \square & \end{array}$$



Deterministic threshold growth model with range 1 box neighborhood, $\theta = 3$, started from the 3 black sites.

Characterization of the shape.

The shape $L = L_1$ is determined by the *Wulff transform*. Imagine that $\bar{\mathcal{T}}$ acts exactly as $\mathcal{T}$ on subsets of $\mathbf{R}^2$. $\bar{\mathcal{T}}$ translates any half-space

$$H_u^- = \{x \in \mathbf{R}^d : \langle x, u \rangle \leq 0\}$$

into

$$\bar{\mathcal{T}}(H_u^-) = H_u^- + w(u) \cdot u, \qquad \text{for some } w(u) \geq 0,$$

and $\bar{\mathcal{T}}(B) \cap \mathbf{Z}^2 = \mathcal{T}(B \cap \mathbf{Z}^2)$. Set

$$K_{1/w} = \cup\{[0, 1/w(u)] \cdot u : u \in S^{d-1}\},$$

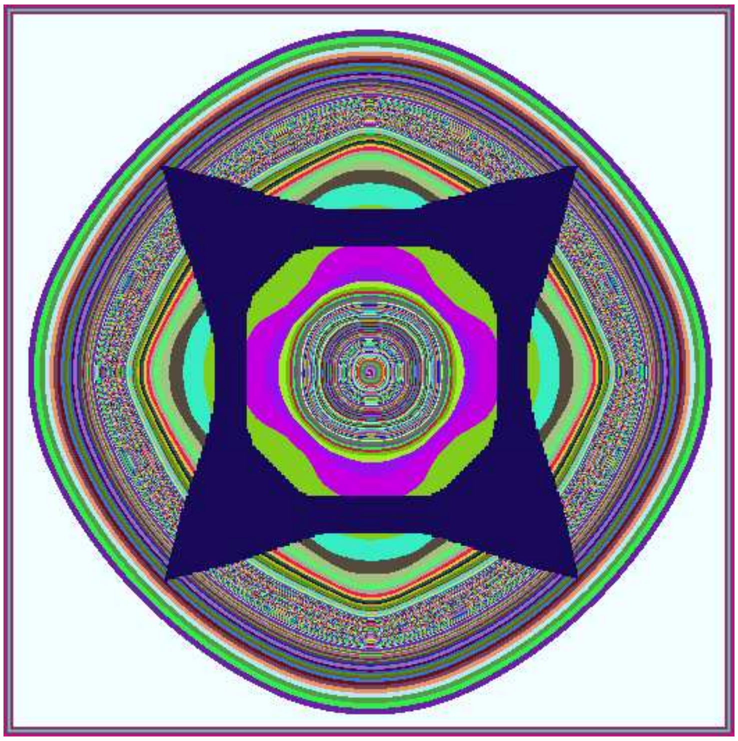
then L is a polygon given by

$$L = K_{1/w}^* = \{x \in \mathbf{R}^d : \langle x, u \rangle \leq w(u) \text{ for every } u \in S^{d-1}\}.$$

The case when $w > 0$ is *supercritical* and is, for box TGM, equivalent to $\theta \leq \rho(2\rho + 1)$.

Smallest growing initial sets.

In the supercritical case, there exists a finite initial set A_0 for which $A_\infty = \mathbf{Z}^2$. For various nucleation questions, the size of smallest such set is of importance. For large ρ, and $\theta \sim \lambda\rho^2$, this size is $\sim \gamma_E(\lambda)\rho^2$. In fact, $\gamma_E(\lambda) = \lambda$ when $\lambda < \lambda_c$ for some $\lambda_c \in (1.61, 1.66)$ (G–Griffeath, 1997).



Initial set (dark blue) which proves that $\lambda_c > 1.61$: this set has size $\theta = 36,760$ for TGM with $\rho = 150$.

Random TGM shapes.

Regularity of growth (Bohman-G, 1999): If $x \in A_t$ is at distance at least $C\rho^4$ from A_0, then there is a set G within distance $\mathcal{O}(\rho^4)$ from x which is occupied and grows forever by itself.

Therefore, the dynamics can be successfully restarted from x, and the shape theorem follows by classic subadditive arguments.

Theorem 1. *If A_0 grows forever, then $A_t/t \to L$ as $n \to \infty$. Here, $L = L_p$ is a bounded convex set with a non–empty interior.*

Proof in a special case.

Assume the *double threshold condition*, that every subset of $\mathcal{N}$ with 2θ sites generates the plane. This is satisfied for box neighborhoods at least when $\theta \le \rho^2$, and otherwise for small enough θ/ρ^2.

For a set X, define its *weight* at time t

$$W_t = \sum_{y \in X \setminus A_t} |(y + \mathcal{N}) \cap A_t|.$$

Furthermore, for any z let $T(z)$ be the time t at which $z \in A_t \setminus A_{t-1}$ and let

$$\Delta_z = |(z + \mathcal{N}) \cap (X \setminus A_{T(z)})| - 1_{\{z \in X\}} |(z + \mathcal{N}) \cap A_{T(z)-1}|.$$

$$\begin{aligned} W_t = W_{t-1} - &\sum_{y \in X \cap (A_t \setminus A_{t-1})} |(y + \mathcal{N}) \cap A_{t-1}| \\ &+ \sum_{y \in X \setminus A_t} |(y + \mathcal{N}) \cap (A_t \setminus A_{t-1})| \\ = W_{t-1} + &\sum_{z \in A_t \setminus A_{t-1}} \Delta_z \end{aligned}$$

and so

$$W_n = W_0 + \sum_{z \in A_n \setminus A_0} \Delta_z.$$

For $x \in A_n$ such that $B_\infty(x, \rho^4) \cap A_0 = \emptyset$ and $r < \rho^4 - 2\rho$, let

$$X = X_r = B_\infty(x, r) \cap A_n,$$
$$B = B_r = (B_\infty(x, r+\rho) \setminus B_\infty(x, r)) \cap A_n.$$

Then $W_0 = W_n = 0$ and so

$$0 = \sum_{z \in X \cup B} \Delta_z \Rightarrow -\sum_{z \in X} \Delta_z = \sum_{z \in B} \Delta_z \leq \rho(2\rho + 1)|B|.$$

Case 1: $\Delta_z \geq 0$ for some $z \in X$. Then, for $t = T(z)$, $|(z + \mathcal{N}) \cap A_{t-1}| \geq \theta$, $|(z + \mathcal{N}) \cap (X \setminus A_t)| \geq \theta$ and $X \setminus A_t \subset A_n \setminus A_{t-1}$, so $|(z + \mathcal{N}) \cap A_n| \geq 2\theta$.

Case2: $\Delta_z \leq -1$ for every $z \in X$. Then $|X_0| = 1$ and

$$\frac{1}{\rho(2\rho + 1)}|X_{\rho i}| \leq |B_{\rho i}| = |X_{\rho i}| - |X_{\rho(i-1)}|.$$

It follows that

$$\left(1 + \frac{1}{\rho(2\rho + 1)}\right)^i \leq |X_{\rho i}| \leq 2\rho^2 i^2 + 1,$$

a contradiction for $i = \rho^{5/2}$.

Random TGM from half–spaces.

Wulff characterization holds as well (G–Griffeath, 2002), although it is not immediate that the half–space velocities $w_p(u)$ exist.

Theorem 2. *Assume that $A_0 = H_u^-$. There exists a deterministic number $w_p(u)$ such that*

$$H_u^- + t(w_p(u) - \epsilon) \cdot u \subset A_t \subset H_u^- + t(w_p(u) + \epsilon) \cdot u,$$

inside the discrete ball of radius t^2, with probability at least $1 - \exp(-ct/\log^2 t)$. (Here, $c = c(\epsilon) > 0$ whenever $\epsilon > 0$.)

Wulff characterization easily follows: if $K_p = K_{1/w_p}$, then $L_p = K_p^*$.

Computational significance: w_p is much easier to compute than L_p.

Sketch of Proof.

For simplicity, assume that $u = e_2$, i.e., initially all sites on or below the x-axis are occupied.

The process is constructed by generating i.i.d. Bernoulli random variables $\xi_{x,t}$ for each $x \in \mathbf{Z}^2$ and each $t = 1, 2, \ldots$ Then let $\mathcal{F}_t = \sigma\{\xi_{x,s} : s \le t, x \in \mathbf{Z}^2\}$.

Let $T_n(i)$ be the first time time (i, n) becomes occupied, $T_n = T_n(0)$ and $\bar{T}_n = T_n \wedge Cn$, for a large enough C so that $P(T_n = \bar{T}_n) \le e^{-cn}$. The main step is the L^∞ bound:

$$(*) \qquad |E(\bar{T}_n \,|\, \mathcal{F}_{s+1}) - E(\bar{T}_n \,|\, \mathcal{F}_s)| \le C \log t,$$

for any $s \le Cn$ and some constant $C = C(p)$.

Let $\mathcal{L}_n$ comprise the space-time sites which influence $\bar{T}_n$. Then of course $|\mathcal{L}_n| \le Cn^3$ and we can assume that the filtration ignores all other sites. At time $s \le Cn$, let ∂A_s consist of all the sites outside A_s which would become occupied if the deterministic dynamics were applied to A_s. Trivially, $|\partial A_s| \le |\mathcal{L}_n|$. When events in $\mathcal{F}_{s+1}$ are revealed, we know which sites in ∂A_s become occupied.

If τ_s is the waiting time *after time* s at which *all* sites in ∂A_s are occupied, then $E(\tau_s) \leq C \log n$. (Here $C \approx -1/\log(1-p)$.) The following inequalities follow from the strong Markov property and monotonicity of the dynamics.

The lower bound follows by assuming the worst case: no sites in ∂A_s get occupied:

$$E(\bar{T}_n \,|\, \mathcal{F}_{s+1}) \leq E(\bar{T}_n \,|\, \mathcal{F}_s) + 1.$$

For the upper bound, assume that $\mathcal{F}_{s+1}$ reveals that all sites in ∂A_s get occupied. Before we know $\mathcal{F}_{s+1}$, we can only assume this happens after time τ_s, and so the dynamics with the additional information is dominated by the one restarted at time $s+\tau_s$. By running this restarted dynamics for $t-s-1$ time units, we obtain

$$E(\bar{T}_n \,|\, \mathcal{F}_s) \leq E(\bar{T}_n \,|\, \mathcal{F}_{s+1}) + E(\tau_s).$$

Hence (*) is proved. Let $a_n = E(T_n)$, $\bar{a}_n = E(\bar{T}_n)$. By Azuma's inequality,

$$(**) \qquad P(|\bar{T}_n - \bar{a}_n| > s) \leq 2\exp(-cs^2/(n\log^2 n)).$$

and, since $|a_n - \bar{a}_n| \leq C$, (**) holds if bars are removed.

Now let T'_n be the first time *all* sites $\{(i,n) : i \leq Cn\}$ are occupied, and T''_n the time when all the sites $\{(i,j) : i \leq Cn, n-C \leq j \leq n\}$ are occupied. By regularity, $E(T''_n - T'_n) \leq C \log n$. Moreover,

$$P(T_n - T'_n \geq s) \leq Cn \exp(-cs^2/(t \log^2 t))$$

and so

$$E(T_n - T'_n) \leq C\sqrt{n} \log^2 n + C \int_{C\sqrt{n}\log^2 n}^{\infty} P(T_n - T'_n \geq s)\, ds,$$

so it follows that $E(T_n - T'_n) \leq C\sqrt{n} \log^2 n$.

By the usual restarting at time $n \geq m$,

$$a_{m+n} \leq a_m + a_n + E(T''_n - T_n) + C,$$

By the deBruijn-Erdös subadditive theorem, a_n/n converges to a finite number a and so does T_n/n, a.s.

Let X_t be the furthest occupied point on the y–axis. From the regularity theorem, at time $t + \epsilon t$, a ball of radius $c\epsilon t$ around $(0, X_t)$ is covered with probability exponentially close to 1. The standard compactness argument now shows that the dynamics a.s. eventually occupies all the sites above $[-t^2, t^2]$ and below the line $y = (1/a - \epsilon)t$, for any ϵ.

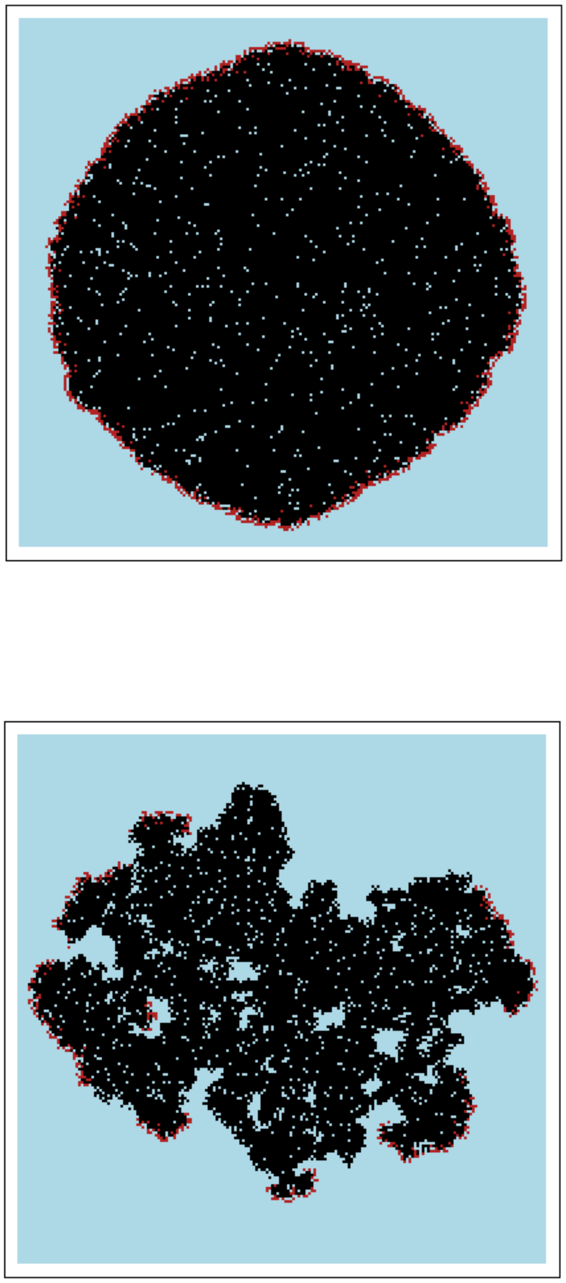
Two open problems.

1. *General monotone CA.* Generalize the regularity theorem to this case.

2. *Shape theorem in a non–monotone case.* Consider the *forest fire CA*, with state space $\{0,1,2\}^{\mathbf{Z}^2}$, and the rules

$$\begin{aligned}
&1 \to 2, \\
&2 \to 2, \\
&0 \to 1, \qquad \text{w.p. } p, \text{ if } \geq \theta \text{ 1's in the nbhd.,} \\
&0 \to 0, \qquad \text{otherwise.}
\end{aligned}$$

Can a shape theorem be proved? (Cox–Durrett, 1988, prove it in the $\theta = 1$ case.)



Forest fire for range 2 box, $\theta = 3$, $p = 0.65$ (top) and $p = 0.52$ (bottom).