Replication in one-dimensional cellular automata

Janko Gravner
Mathematics Department
University of California
Davis, CA 95616
e-mail: gravner@math.ucdavis.edu

Genna Gliner
Mathematics Department
University of California
Davis, CA 95616
e-mail: grgliner@ucdavis.edu

Mason Pelfrey
Mathematics Department
University of California
Davis, CA 95616
e-mail: mtpelfrey@ucdavis.edu

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Abstract. In cellular automata (CA), replication is the ability to indefinitely generate copies of a finite collection of patterns, starting from finite seeds. A transparent feature of additive CA, replication mechanisms are less clear in the absence of additivity; this paper investigates such dynamics through several examples. For the *1 Or 2* rule and its generalizations, replication is inevitable and we investigate self-organization properties. In the *Perturbed Exactly 1* rule we study frequency of replicators and the new phenomenon called quasireplication. The last CA is the *Extended 1 Or 3* rule, which allows for replication on different backgrounds. We employ a mixture of rigorous and empirical techniques.

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1 Introduction

During the evolution of many simple cellular automata (CA) rules, especially those related to the *Game of Life*, the following phenomenon is frequently observed: "a configuration of occupied sites makes copies of itself, then the copies make copies of themselves, and these copies move toward one another. [...] When the innermost copies collide, they annihilate, [while] the outermost [ones] continue to reproduce. This pattern repeats, ad infinitum" [**Eva2**]. The resulting space-time picture is sometimes called *fractal* [**CD**] or *nested* [**Wol3**], but we prefer the more descriptive term *replication* [**Eva1**, **Eva2**]. Traditionally associated with additivity [**CD**], this type of behavior occurs for no a priori reason in numerous CA, in one and two dimensions [**Epp**]. In the present paper, we focus on selected one-dimensional examples.

To illustrate our main ideas, two CA (whose rules are defined later in this section) were run from a small random seed. Fig. 1 depicts the resulting space-time configurations; the time axis in all our pictures and descriptions is oriented *downward*, as is common in this field. In each case of Fig. 1, triangular regions appear, either empty or filled with a simple periodic pattern. Is this a beginning of a recursive behavior, with larger and larger regular regions? What kind of periodic patterns can be generated? What mechanisms govern the initial self-organization phase? How typical is such a dynamics for a given CA? Much of the rest of the paper is devoted to precise formulations of such questions, and to techniques for obtaining at least partial answers.

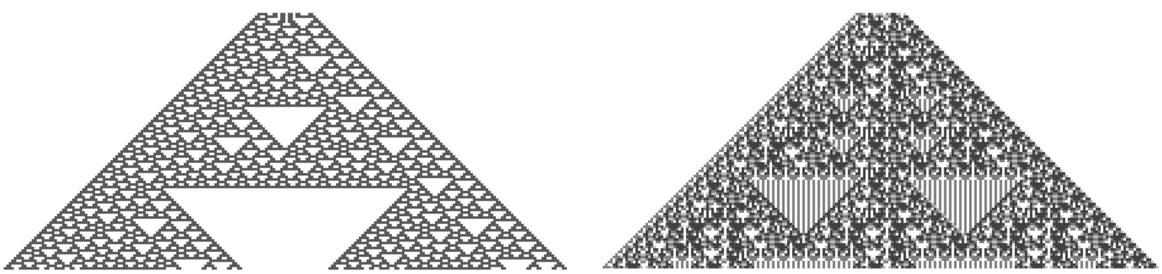

Fig. 1. Two replicator examples, both at time 100. Left: *Quota* with $\theta = 2$, started from 1111001101101101100011; right: *Extended 1 Or 3*, started from 2221032011102133101.

We present a basic replication setup in Section 2, defining the ingredients that go into the description of a replication scheme. Given the diverse circumstances in which replicators appear [**Eva1**, **Eva2**, **Epp**], it seems unlikely that one could formulate either wide-ranging necessary and sufficient conditions on existence of replicators, or estimate the likelihood of their appearance from random initial states independently of the details of a particular rule. Instead, we develop a list of issues one might look at when presented with a CA rule capable of replication. We also do not close the book on any of the presented examples, and conclude the paper with an inventory of interesting open problems.

We call a *configuration* a finite or infinite sequence of elements from S, where $S = \{0, 1, \ldots, s-1\}$ is a finite set of *states*. When the sequence is doubly infinite, a configuration gives each site in $\mathbb{Z}$ either the *empty* state 0, or one of $s-1$ *occupied states*. We will assume that time increases in discrete steps, $t = 0, 1, 2, 3, \ldots$, and that configurations ξ_t evolve by a CA rule. A configuration is *finite* if it has only finitely many occupied sites; the *initial* configuration ξ_0 will be typically assumed finite, and will in this case often be referred to as a *seed*. As we follow the usual convention [**GG4**] that the state of a site, when unspecified, is assumed to be 0, we may give a finite configuration as a finite sequence of states. Finally, when $S = \{0,1\}$ we identify the configuration ξ_t with its set $\{\xi_t = 1\}$ of occupied sites at time t. In all examples we present, the quiescent (all 0's) state is mapped into itself; to avoid the trivial case we will always assume that a seed is non-quiescent.

We are interested in cases when a CA *replicates*, that is, makes copies of a finite collection of finite configurations, called *replicating elements*, indefinitely. Such dynamics may occur for all, or only for some, seeds. As in [**GG4**], we identify the seed with the attractor, calling it a *replicator* if it leads to replication regardless of whether the seed is among the replicating elements.

The easiest to study are *additive* CA, which for our purposes have $S = \{0,1\}$ and are given by a finite (*neighborhood*) $\mathcal{N} \subset \mathbb{Z}$. Then the CA λ_t is given for $t = 0, 1, \ldots$ by addition modulo 2 over the neighborhood given by $\mathcal{N}$:

$$\lambda_{t+1}(x) = \sum_{k \in \mathcal{N}} \lambda_t(x+k) \bmod 2.$$

The most basic example is known as *Rule 90* [**Wol1**] and has $\mathcal{N} = \{-1, 1\}$. It has been long known that, at large enough times of the form 2^n, the occupied set consists of two identical copies of the seed (which, we emphasize again, is finite), separated by 0's. Another additive rule, *1 Or 3* or *Rule 150*, in which $\mathcal{N} = \{-1, 0, 1\}$, behaves in the same way, except that the number of copies is now three; in fact, every additive CA replicates any seed, and the number of copies at large dyadic power times equals cardinality of $\mathcal{N}$. Additive rules commute with addition modulo 2 and seem to be the only class of CA rules amenable to general mathematical theory (e.g., [**Wol1**, **CD**, **Wil1**, **Wil2**, **FLM**, **HPS**]), thus they play a role analogous to linear dynamical systems, and are for this reason sometimes called *linear*.

As we will see more formally in Section 2, a CA started from a particular seed replicates if it simulates an additive CA started from a single 1 at the origin. This is a very weak version of the important notion of intrinsic simulation, which demands that one CA is able to simulate another started from any initial configuration (see [**Oll**] and subsequent work of the same author). We also remark that, if a seed is a replicator, its trajectory can be efficiently predicted ([**Moo**]).

We now introduce the examples of nonadditive CA considered in this paper. These have all previously appeared in the literature, and are selected for connections to other interesting dynamics, such as two-dimensional CA growth [**GG1**, **GG2**, **GG3**] and coupled logistic-type maps [**GG4**], and especially for their simplicity. In particular, each of our CA uses ether a range 1 or a range 2 neighborhood, i.e., the neighborhood of an integer point x is either $\{x-1, x, x+1\}$ or $\{x-2, x-1, x, x+1, x+2\}$.

We begin by the rule we consider the prototypical nonadditive dynamics [**GG4**], namely the *Exactly 1* CA (often called *Rule 26* [**Wol1**]), in which x becomes occupied at time $t+1$ if and only it has exactly 1 occupied nearest neighbor at time t:

$$\xi_{t+1}(x) = 1 \iff |\xi_t \cap \{x-1, x, x+1\}| = 1.$$

(The vertical bars denote cardinality.) Replication is common in this CA among small seeds. For large seeds, chaotic evolution is by far the likeliest, and periodic seeds also exist. The paper [**GG4**] contains a detailed study including many replication examples.

The next rule is also quite well-known. We call it the *1 Or 2* CA, but it is also known as *Rule 126* [**Jen3**, **Jen4**, **GG3**]. It has binary states and its rule mandates that the state of x is occupied at time $t+1$ if and only if either one or two of its range 1 neighbors of x are occupied at time t:

$$\xi_{t+1}(x) = 1 \iff |\xi_t \cap \{x-1, x, x+1\}| \in \{1, 2\}.$$

This rule is not additive; nevertheless it replicates for every seed, and it always has essentially a single replicating element. We call such CA *quasiadditive*, and two additional such examples (which are not additive) have been studied [**Jen2**, **Jen3**, **Jen4**]. The replicating element can be very different from the seed, due to a long *onset* time before the replication commences. We will study the distribution of onset times in Section 3.

We also prove that two natural generalizations of Jen's rules to range 2 are both quasiadditive. These are *Quota* rules [**CD**] with a threshold parameter θ and stipulate that a point becomes occupied if it sees at least θ 1's *and* at least θ 0's in its range 2 neighborhood:

$$\xi_{t+1}(x) = 1 \iff |\xi_t \cap \{x-2, x-1, x, x+1, x+2\}| \in [\theta, 5-\theta].$$

As described in Section 4, these have an extra self-organizing period before they enter the Jen regime. When $\theta = 1$ this period is trivial and lasts a single time step, but when $\theta = 2$ a Lyapunov function drives the dynamics toward a configuration with sufficient regularity.

Next we describe a CA we call *Perturbed Exactly 1*, introduced in [**BP**]. This rule also has binary states but now x is occupied at time $t+1$ if and only if at time t either (1) it has exactly 2 occupied sites among its five nearest neighbors or (2) the single occupied site among its five nearest neighbors is positioned among the three nearest neighbors:

$$\begin{aligned} \xi_{t+1}(x) = 1 \iff & |\xi_t \cap \{x-2, x-1, x, x+1, x+2\}| = 2 \text{ or} \\ & |\xi_t \cap \{x-1, x, x+1\}| = 1 \text{ and } \xi_t(x-2) = \xi_t(x+2) = 0. \end{aligned}$$

First few experiments with small seeds, as well as the account in [**BP**], suggest that replication always happens for *Perturbed Exactly 1*, but this is not the case. However, as we demonstrate in Section 5, replication is indeed quite common, although, as is *Exactly 1*, this CA is capable of chaotic behavior. Even more interesting, and challenging to study, is a new type of behavior we call *quasireplication*, whereby the occupied set within space-time is fractal, in the appropriate limit, even in the absence of replication (see Section 6).

In the examples presented so far, all the known instances of replication proceed on an empty background, but nonzero periodic backgrounds, called *ethers*, are also possible. *Rule 186* [**Wol3**], for instance, always replicates on the fully occupied ether (which can be easily proved because its evolution is the"photographic negative" of the one for *Rule 146* studied in [**Jen4**]). Perhaps the simplest example which admits both the zero ether and a nonzero one is the *Embossed Triangles* CA, introduced by M. Wójtovicz in [**Woj**]. In this range 2 rule, a 1 survives by contact with two or three 1's (including itself), and a 0 changes to 1 by contact with two, three, or four 1's:

$$\begin{aligned}\xi_{t+1}(x) = 1 \iff & \xi_t(x) = 1 \text{ and } |\xi_t \cap \{x-2, x-1, x, x+1, x+2\}| \in \{2,3\} \\ & \text{or } \xi_t(x) = 0 \text{ and } |\xi_t \cap \{x-2, x-1, x, x+1, x+1\}| \in \{2,3,4\}.\end{aligned}$$

Both *Quota* and *Embossed Triangles* fit into context of *Larger Than Life* CA [**Eva1**], which seem to be a particularly fertile ground for replication.

To our knowledge, the best case to investigate multiple ethers is our last rule, which requires a little motivation. One of the most interesting two-dimensional solidification CA is *Box 13* [**GG3**]. In this dynamics on $\mathbb{Z}^2$ with binary states, an occupied site always stays occupied, while an empty state becomes occupied if it has either 1 or 3 already occupied sites in its Moore neighborhood, i.e., among its nearest eight sites. Assume that all of the initially occupied sites are initially on or below the x-axis, and that the x-axis contains at least one occupied site. Then, for any CA that uses the Moore neighborhood, the configuration at time t on the line $y = t$ only depends on the configuration on the line $y = t-1$ at time $t-1$, and this dependence defines the *extreme boundary dynamics (EBD)* for the CA.

For the *Box 13* solidification, EBD is the additive *1 Or 3* rule; thus, by analogy to *Hex* [**GG2**] or *Diamond* [**BDR**] rules, one expects at first that this solidification CA is amenable to complete analysis. However, a new problem appears: the web of occupied sites generated by the EBD "leaks," and therefore fails to divide the lattice into independent regions with a renewal structure. A more successful approach involves keeping track of *two* extreme lines, $y = t$ and $y = t-1$. The resulting one-dimensional CA, which we call *2-level EBD* for *Box 13*, or *Extended 1 Or 3*, is no longer additive. For a given x, we code the four occupation possibilities of $(x, t-1)$ and (x, t) at time t as $00 = 0$, $01 = 1$, $10 = 2$ and $11 = 3$ to obtain a CA with $S = \{0, 1, 2, 3\}$, and the following convoluted rule. First compute

$$\begin{aligned} c_1 &= (\xi_t(x-1) + \xi_t(x) + \xi_t(x+1)) \bmod 2, \\ c_2 &= \left\lceil \frac{\xi_t(x-1)}{2} \right\rceil + \left\lceil \frac{\xi_t(x)}{2} \right\rceil + \left\lceil \frac{\xi_t(x+1)}{2} \right\rceil, \end{aligned}$$

and then let

$$\xi_{t+1}(x) = \begin{cases} c_1 + 2, & \text{if either } \xi_t(x) \bmod 2 = 1 \text{ or } c_2 \in \{1,3\}, \\ c_1, & \text{otherwise.} \end{cases}$$

Replication in *Extended 1 Or 3* is common, on a zero ether or on one of many other ethers (see Section 7). In fact, an overwhelming proportion of large seeds seem to replicate. On the other hand, we will demonstrate that a simple seed is a non-replicating quasireplicator. There is some empirical evidence for existence of a much more complex evolution with different properties than, say, the *Exactly 1* chaos [**GG4**]. However, some seeds take an extraordinarily long time to organize and the possibility of ethers with enormous periods cannot be eliminated; thus conjectures of asymptotic behavior based on even millions of updates are precarious.

The above cautionary note is important, but the role of the computer experimentation in present investigations cannot be overstated. Many of our results were first conjectured through computer experimentation, using *MCell* [**Woj**], or one of our many ad hoc programs. Reading this paper would be greatly facilitated by seeing some of the described dynamics in action, thus we will maintain a collection of experiments related to this project at [**GGM**].

2 Preliminaries

We start by listing a few common conventions. We will call a finite configuration's left endpoint its *placement*. This is often important, as we build configurations from appropriately placed finite pieces. If left unspecified, the placement is at the origin. Another useful rule is that the state of a site, when not given, is 0. As is customary we put a configuration η in a superscript to indicate that it is used as the initial configuration, e.g., ξ_t^η indicates that ξ_t evolves from $\xi_0 = \eta$.

Assume that π is a doubly infinite configuration with *spatial period* σ, i.e., $\pi(x+\sigma) = \pi(x)$ for every x. Assume also that it is periodic with *temporal period* τ for the CA ξ_t, i.e., $\xi_{t+\tau}^\pi = \xi_t^\pi$ for all $t \geq 0$. Then we call π a *periodic solution*; we will assume that both periods σ and τ are minimal. We may give such a configuration by its repeating segment; for example, 10^∞ stands for $\ldots 0101010\ldots$, a fixed point (i.e., $\tau = 1$) of *Exactly 1*. We will not study periodic attractors in this paper (see [**GG4**] for an extensive investigation in the *Exactly 1* case), but instead merely use periodic solutions as backgrounds for interaction between replication elements; accordingly, we will call them *ethers*.

A *replicator rule* consists of the following ingredients:

1. an ether π;
2. a finite nonempty set $\mathcal{K}$ of finite configurations that comprises the *replicating elements*;
3. a finite set $\mathcal{N} \subset \mathbb{Z}$, the neighborhood for the additive rule λ_t started from a single 1; and
4. a function $\texttt{successor} : (\mathcal{K} \cup \{0_\infty, 0_\pi\})^{\mathcal{N}} \to \mathcal{K}$. If $n = |\mathcal{N}|$, we may represent $\texttt{successor}$ as a n-ary operation which we denote by $\oplus$.

A finite initial configuration ξ_0 is a *replicator* for a CA ξ_t if, after a proper placement, there exist a replicator rule (π, $\mathcal{K}$, $\mathcal{N}$, $\texttt{successor}$), and $t_0 \geq 0, n_0 \geq 0$, so that the configurations ξ_t at times $t = t_0 + 2^{n_0}(k-1)$, $k = 1, 2, \ldots$, satisfy the following:

- for every x such that $\lambda_k(x) = 1$ there is a copy of a $K_{k,x} \in \mathcal{K}$ placed at $2^{n_0}x$;
- the remaining sites consist of two infinite intervals, filled with 0's, and a number of bounded intervals, each of which contains a segment of the ether π;
- each placed replicating element is distinguishable from the ether, i.e., if $[a+1, b-1]$ contains a placed element, then $[a, b]$ is *not* a segment of the ether;
- $K_{k,x}$ is given by $\texttt{successor}(K_{k-1,y} : y \in x + \mathcal{N})$; if $\lambda_{k-1}(y) = 0$, then $K_{k-1,y}$ is interpreted as 0_π if $y \in [k \cdot \min(\mathcal{N}), k \cdot \max(\mathcal{N})]$ and as 0_∞ otherwise.

We will assume, without loss of generality, that a fixed replicating element K, when in contact with the ether π from either side, encounters π in the same spatial phase: the first σ states of π are the same on either side of every occurrence of K.

Thus all $K_{k,x}$ are determined by the initial elements $K_{1,x}$, $x \in \mathcal{N}$, and successive applications of the succession rule `successor`. Also note that the replicating elements may be replaced by their successors by the original CA rule, so $\mathcal{K}$ is not unique. In our examples and in Section 3, we will take a $\mathcal{K}$ with the smallest cardinality and assume not all of its elements can be shortened (while still remaining a set of replicating elements). Then we will assume that the *onset time* t_0 and the *replication time* 2^{n_0} are minimal (note that selection of t_0 and the two elements at that time determines n_0). The final, and very important, remark is that the specification of the rule `successor` may not be complete; then one has to give an argument that the missing interactions never happen.

We call ξ_0 a *maternal* replicator if the ether is the zero configuration, and there exists a configuration K so that any configuration in $\mathcal{K}$ equals K, possibly with 0's appended at either end. Any replicator which is not maternal is *fraternal*.

We call a CA for which every seed is a maternal replicator *quasiadditive*. Every additive CA is quasiadditive; this well-known folk result is easy to prove [**CD**]. Not too many quasiadditive CA that are not additive are known, but three are introduced in [**Jen3**]: *1 Or 2*, *Rule 18*, and *Rule 146*. Existence of a fraternal indicator is thus a sign of an essential nonadditivity in a CA rule.

We remark that the definition of a replicator can be substantially simplified when the ether is zero and λ_t is *Rule 90*, that is, when $\mathcal{N} = \{-1, 1\}$. See [**GG4**] for the definition in that case, and for many illustrative examples of maternal and fraternal replicators for *Exactly 1* rule. For higher-dimensional emulation of additive rules, see [**Eva2**].

It is important to realize that one could formulate a condition to verify that an initial state is a replicator after only finitely many replications, when all possible interactions have taken place; see [**GG3**] for some examples, and we give another below. Thus in every particular instance a computer can be used to verify that a seed is replicator. Invariably, the details of such verifications do not translate well from the computer screen to text, hence they are largely omitted from the paper.

Fig. 2 depicts two illustrative examples. The *Extended 1 Or 3* one, being maternal, has a very simple description. (See Theorem 6 in Section 7 for the general result that covers this example.) The *Embossed Triangles* one is more complicated and for once we give a complete description of its replicator algebra. There are 12 replicating elements. Six of them,

$$\begin{aligned} A &= 1110110111110000, \\ B &= 1110111100000011, \\ C &= 1110100011001100, \\ D &= 1111001111110000, \\ E &= 0011110011001100, \\ F &= 1100111100000011, \end{aligned}$$

inhabit the left portion of the space-time configuration. The remaining six, distinguished by the superscript R, inhabit the right portion, and are mirror images, with appropriate segments of the ether (ten sites in each case) added to make their placement correct. The only left-right interactions are between A and A^R at the beginning, and between D and D^R later on, both of course giving 0_π. Here is the list of other interactions, denoted by the noncommutative operation $\oplus$:

$$\begin{aligned}
&A \oplus 0_\pi = E, \quad 0_\infty \oplus A = B,\\
&B \oplus 0_\pi = D, \quad 0_\infty \oplus B = C,\\
&C \oplus 0_\pi = F, \quad 0_\infty \oplus C = A,\\
&D \oplus 0_\pi = E, \quad 0_\pi \oplus D = F,\\
&E \oplus 0_\pi = F, \quad 0_\pi \oplus E = D,\\
&F \oplus 0_\pi = D, \quad 0_\pi \oplus F = E,\\
&A \oplus F = B \oplus E = C \oplus D = D \oplus F = E \oplus D = F \oplus E = 0_\pi.
\end{aligned}$$

The last verification, $0_\pi \oplus D = F$, occurs at time $248 = t_0 + (15-1) \cdot 2^{n_0}$, and one can prove by induction that no other interactions ever occur. Therefore, it is at time 248 that we can be truly certain that this dynamics emulates *Rule 90*.

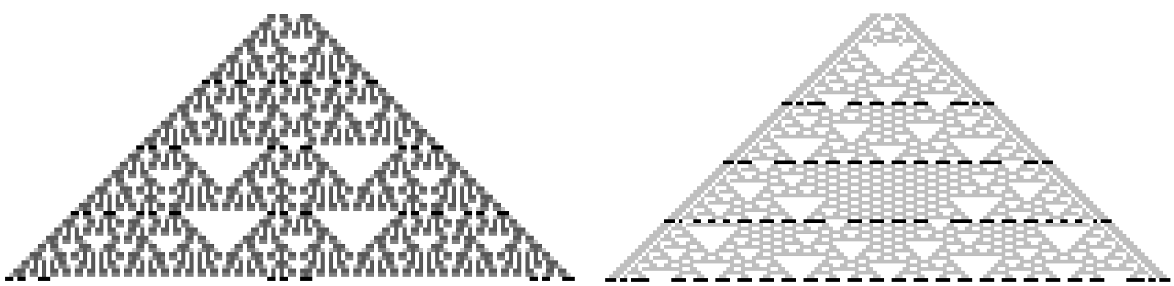

Fig. 2. Two replication examples; in each case, the configurations at the onset time and at three multiples of the replication time thereafter are highlighted in black. Left: *Extended 1 Or 3* from $A = 33033000333$, a maternal replicator, in which $\mathcal{K} = \{A\}$, $t_0 = 2^{n_0} = 16$, and $\mathcal{N} = \{0, \pm 1\}$; the states 3, 2 and 1 are in progressively lighter shades of gray. Translations of A occur at positions given by the locations of 1's, multiplied by 16, in the additive *1 Or 3* CA. Right: *Embossed Triangles* from 1110110111, with $\mathcal{N} = \{\pm 1\}$, ether $\pi = 111100^\infty$ with $\sigma = 6$, $\tau = 2$, $\mathcal{K}$ and `successor` described in the text, and $t_0 = 24$, $n_0 = 4$. The additive CA is thus *Rule 90* and the elements (from left to right) at time t_0 are A and A^R; then B and B^R at time $t_0 + 2^{n_0}$; then C, D, D^R, and C^R at time $t_0 + 2 \cdot 2^{n_0}$; and then A and A^R at time $t_0 + 3 \cdot 2^{n_0}$.

Define the space-time occupied set

$$A_t = \{(s, x) : 0 \le s \le t, \xi_s(x) > 0\} \subset \mathbb{Z}^2.$$

Assume the rescaled subsequence

$$\tilde{A}_n = \frac{1}{2^n} A_{2^n} \subset \mathbb{R}^2$$

converges to a limit set $\tilde{A}_\infty$ in the Hausdorff metric. Then we call $\tilde{A}_\infty$ a *Willson limit* of the CA; clearly, it may depend on the seed. We call a seed ξ_0 a *quasireplicator* if $\tilde{A}_\infty$ exists, and has its Hausdorff dimension in the open interval $(1,2)$. The following theorem follows immediately from [**Wil2**].

Theorem 1. *Every replicator with zero ether is a quasireplicator.*

In Sections 6 and 7 we give examples of seeds that are provably quasireplicators but not replicators.

Denote $A_\infty = \cup_t A_t$, and let μ_ϵ be ϵ^2 times the counting measure on $\epsilon \cdot A_\infty$. That is, for any Borel set B, $\mu_\epsilon(B) = \epsilon^2 \cdot |B \cap (\epsilon A_\infty)|$. Fix an open set $W \subset \mathbb{R}^2$. We say that A_∞ has density ρ on W if for any continuous function f, which is compactly supported inside W,

$$\int f \, d\mu_\epsilon \to \rho \cdot \int f \, dx,$$

as $\epsilon \to 0$. Our default choice of W will be the wedge $W = \{(x,t) : t \min(\mathcal{N}) < x < t \max(\mathcal{N})\}$. It is easy to see that a quasireplicator has density 0, whereas a replicator with a nozero ether has a strictly positive density (and hence the Willson limit of dimension 2). The same property, albeit perhaps on a smaller wedge, also holds for apparently ubiquitous chaotic seeds [**GG4**], although it has not been rigorously proved that any seed is chaotic for any CA.

In our replicator definition we have assumed a single ether. A more general definition would allow *mixed* replicators which allow for any ether from a finite collection in intervals between replicators. We could also allow *stitches*, bounded perturbations of the ether near, say the origin, which do not effect interaction between the elements. Such cases are common but complicate the discussion without adding anything new; nevertheless, we provide two examples of mixed replicators in Fig. 3. A much more substantial generalization would allow for an arbitrary group in place of $\mathbb{Z}_2$, and this would certainly be necessary for a complete study of CA replication.

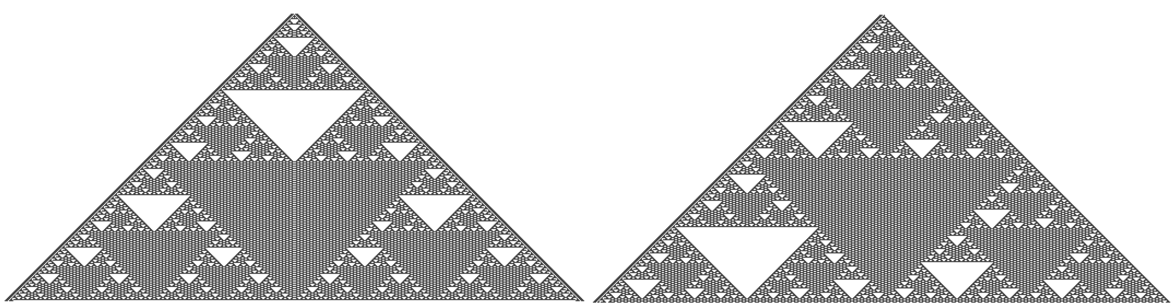

Fig. 3. *Embossed Triangles* at time $t = 262$, from 1110111 (left) and 11101 (right), both mixed examples with two ethers, $\pi_1 \equiv 0$ and $\pi_2 = 111100^\infty$. Note that the Willson limit does not exist for the left example (whose ether is also stitched).

For a given property of evolution, i.e., a set of space-time configurations $\mathcal{P} \subset 2^{S^{\mathbb{Z}\times\mathbb{Z}_+}}$, we define its *entropy* $h = h_{\mathcal{P}}$ as follows. Call a *seed of length* $n+2$ any configuration in $[0, n+1]$, with $\xi_0(0) \neq 0$, $\xi_0(n+1) \neq 0$ (and 0's in $[0,n+1]^c$). Let $N_{\mathcal{P},n}$ be the number of seeds ξ_0 of length $n+2$ that make $(x,t) \mapsto \xi_t(x)$ a member of $\mathcal{P}$. Then

$$h = h_{\mathcal{P}} = \liminf_{n\to\infty} \frac{1}{n} \log_s N_{\mathcal{P},n}.$$

This quantity measures the amount of choice per position one has in the design of a long seed whose evolution satisfies a given property, and has little to do with space-time entropies of early CA research [**Wol1**, **Wol2**]. Note that we normalize so that without any constraints, i.e., $\mathcal{P} = 2^{S^{\mathbb{Z}\times\mathbb{Z}_+}}$, $h = 1$. We often restrict the n's in the limit to odd or even subsequences, which we indicate in the superscript, h^o or h^e.

3 Onset times in *1 Or 2*

Throughout the paper, we will call a *block* a maximal contiguous interval of a single state, with its *length* equal to its number of sites.

The *1 Or 2* CA is perhaps the simplest quasiadditive one [**GG3**], thus it presents an opportunity to take a closer look at the onset times t_0 and the resulting replicating elements. As we will explain below, there are simple algorithmic definitions for both, which facilitate an empirical analysis. Any rigorous confirmation of our conclusions would have to proceed through understanding of the annihilating diffusion of odd blocks [**HC**, **EN**] and statistical properties of configurations up to the additivity time defined below. This still presents a very substantial challenge.

We now describe the key features of the *1 Or 2* evolution [**Jen3**, **GG3**]. Every block has a unique successor, either the block of 0's of length diminished by 2 immediately below, or, for blocks of length 1 and 2, a larger block of 1's immediately below. New blocks appear, but they are always of even length, and the odd blocks pairwise annihilate when their successor is the same block. There is a finite time, which we call the *additivity time* t_a, the least time at which at most one block of odd length is left. For $t \geq t_a$, the dynamics is conjugate to *Rule 90*: if we enlarge the odd block (if any) by insertion of a site of the same state, the configuration evolves by *Rule 90* thereafter, and the state ξ_t is obtained by deletion of the extra site from the successor of the modified block. Maternal replication easily follows [**GG4**].

It is well known, and easy to show, that *Rule 90* is injective but not surjective on finite configurations. Thus every finite configuration A has a unique predecessor $P(A)$ of shortest length. We also denote by $J(A)$ the *Jen's conjugacy* map, which inserts a site of the same state into every odd block of A.

Assume we start ξ_t from a seed of length $n + 2$. By parity, if n is even, there are no odd blocks at time t_a and the dynamics is *Rule 90* thereafter. If n is odd there is a single odd block, which is at replication times the middle block of 0's [**GG4**].

Proposition 3.1. *If n is even, the shortest representation of the only replicating element is $P(\xi_{t_a})$. If n is odd there are two replicating elements: $P(J(\xi_{t_a}))$ and the same one with a prefixed 0.*

In either case, t_0 is the first time $t \geq t_a$ at which ξ_t consists of two identical configurations placed at distance 2^{n_0+1} (n even) or $2^{n_0+1} - 1$ (n odd) for some $n_0 \geq 0$. This determines the replication time 2^{n_0}.

Proof. It is clear that the claimed elements replicate, and a shorter element would yield a shorter

Rule 90 ancestor of $J(\xi_{t_a})$. If n is even, then $2^{n_0-1} = t_0 - t_a$ is the first power of 2 greater or equal to the length $n_0 + 2 + 2t_a$, as this is the first time the additive dynamics "separates" the two copies of ξ_{t_a}. If n is odd, however, one needs to wait until the successor of the odd block in ξ_{t_a} is the middle block of 0's between the two elements, and this happens precisely at the claimed time. See Fig. 4 for an illustration. □

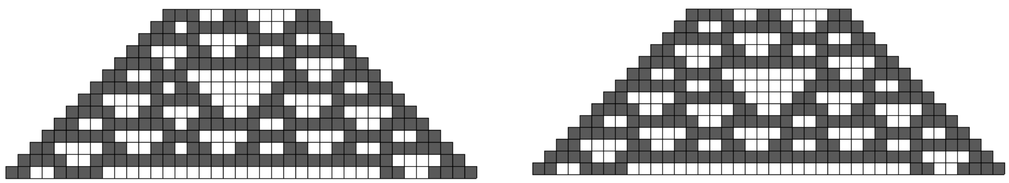

Fig. 4. *1 Or 2* from $A = 1110011000011$ (left) and $J(A)$. Here $P(J(A)) = 1100111$. Observe that $t_a = 0$ for both, that $t_0 = 5$ on the right, but $t_0 = 13$ on the left as the odd block at $t = 5$ is not the middle block of 0's.

Using Proposition 3.1, we computed the distributions of t_a and t_0 for small seed lengths, and the resulting histograms for $n = 25$ are presented in Fig. 6. It is clear that the onset times are even for low n highly clustered. The highest peaks in the histogram for t_a are approximated by powers of 2. These are the times when odd blocks annihilate each other on the upper borders of large triangles of 0's, which are bound to appear as the dynamics is conjugate to *Rule 90* when the number of odd blocks is constant [**Jen3**]. This is the prevailing annihilation mechanism — within nearly chaotic regions of small vacant triangles the odd blocks undergo much slower diffusive annihilation [**HC**]; see also Fig. 5 for an example. (The much smaller peaks at the tail of the distribution have to do with fine details of annihilation process and predecessor distribution and are harder to quantify.)

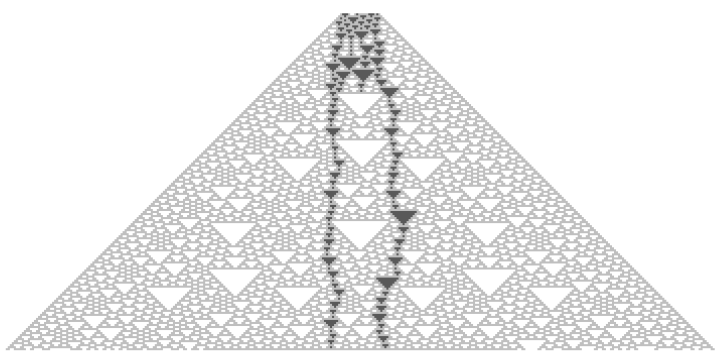

Fig. 5. Annihilation of odd blocks for 10001001011111000101, a configuration which has maximum $t_a = 166$ for $n \leq 18$. Only odd blocks of 0's are emphasized and the final time shown is t_a. In this case $t_0 = 422$, exactly the time obtained by the separation rule.

In most cases, a data point at $t = p$ in the t_a-histogram is translated at least approximately, into a data point in the t_0-histogram by the following *separation rule*. Let 2^k be the smallest power of 2 such that $2^k > n + 2 + p$. This is the smallest time at which the additive dynamics separates two copies of the "fixed" configuration at time p, which has length $n+2+p$. This yields a data point $p+2^k$ in t_0-histogram. (How accurate is this rule depends on the location of the odd interval in the configuration at time p and on its predecessors.) For example, the t_a-peak at 17 in Fig. 6 yields a t_0-peak at $17+64=81$. The times between peaks in t_a-histogram are also subject to the separation mechanism, which thus leads to a more clustered nature of t_0-histogram.

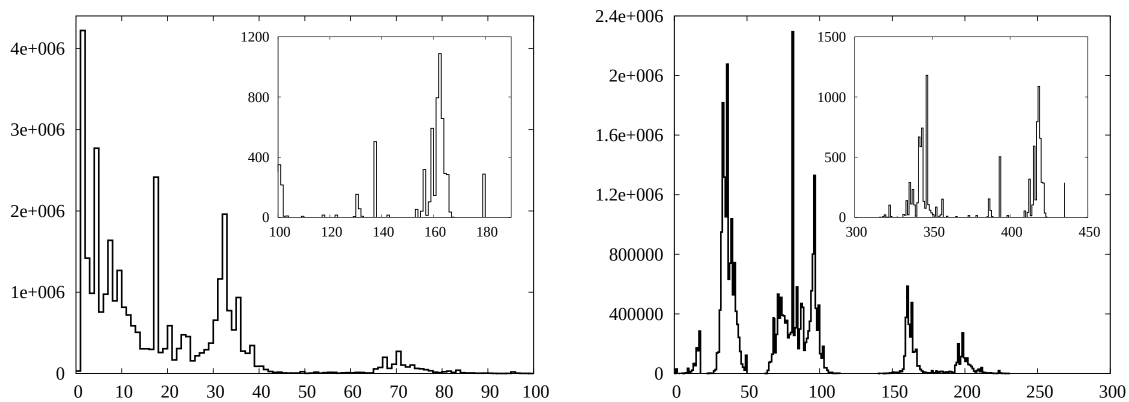

Fig. 6. Histograms for seeds of length 27 ($n=25$). Left: additivity time t_a; right: onset time t_0. The insets show tails of the distributions on a smaller y-scales (the rightmost data points are $\max t_a = 179$ and $\max t_0 = 435$).

As additive dynamics and annihilating diffusion of odd blocks interact, the location of t_a-peaks diverges significantly from powers of 2 for larger n. In addition, there is noticeable difference between seeds of odd and even length, as the former are more capable of having smaller additivity times. For a clearer picture, we restrict to even n, and provide the evolution of empirical t_a-histograms, based on $50n$ samples, for even n from 50 to 250; the results are shown in Fig. 7. We observe that there are three or four prominent peaks; when n is around a power of 2 the peak at the smallest time gradually lowers while the one at the highest time starts rising.

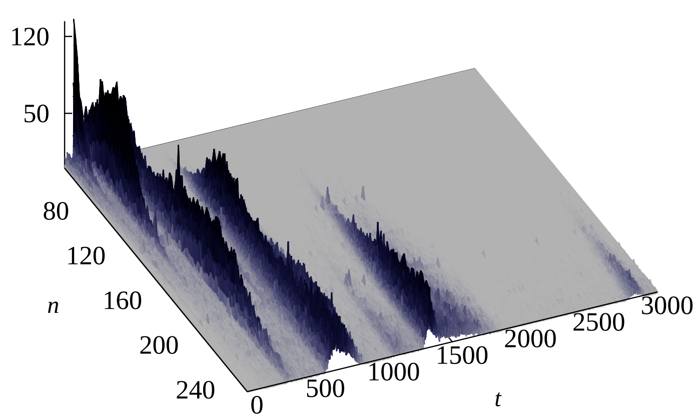

Fig. 7. Evolution of t_a-histograms.

4 Maternal replication in *Quota*

We begin by noting the useful *complement property* of *Quota* CA: if $\eta^c = 1 - \eta$, then $\xi_t^{\eta^c} = \xi_t^{\eta}$ for $t \geq 1$.

Theorem 2. *Quota with $\theta = 1$ is a quasireplicator.*

Proof. Step 1 (self-organization). At time 1, and hence afterward, all blocks of 1's are of length at least 4.

To prove this, assume that $\xi_1(x) = \xi_1(x+4) = 0$ at time 1. This means that ξ_0 is either all 0 or all 1 on $[x-2, x+2]$ and the same is true for $[x+2, x+6]$, and consequently on $[x-2, x+6]$. Thus $\xi_1(y) = 0$ for $x \leq y \leq x+4$.

Next four steps deal with the *Jen regime*, whereby irregular blocks (in this case those whose lengths are not 0 modulo 4), diffuse and pairwise annihilate. The proofs are straightforward checks [**Jen4**].

Step 2 (block succession). Assume that 1's in ξ_0 occur only in blocks of length at least 4. Then every block has a unique *successor*, a block immediately below it. A block of 0's, or that of 1's, of length $\ell \geq 5$ shrinks into a block of 0's of length $\ell - 4$. A block of 0's, or that of 1's, of length $\ell \leq 4$ has as a successor a block of 1's of length at least 5.

Step 3 (additive configurations). Assume that all blocks of ξ_0 are of length divisible by 4. Then this is true for all t. Moreover, assume also that $\xi_0(0) = 1$ and $\xi_0(-1) = 0$, and let $\lambda_t(x) = \xi_{2t}(4x)$. Then λ_t evolves as *Rule 90*.

Step 4 (Jen conjugacy). Keep the assumption from Step 2. For every configuration ξ, define $J(\xi)$ to be the configuration that elongates every block of length ℓ to the length $4 \cdot \lceil \ell/4 \rceil$. If the number of blocks of length not equal to 0 mod 4 is constant during time interval $[0, t_1]$, then $J(\xi_t)$ is the same as ξ_t evolved from $J(\xi_0)$.

The proof is now concluded by an argument exactly like that for Lemma 3.5 in [**GG4**]. $\square$

We now turn to the more interesting $\theta = 2$ case. We say that a seed ξ_0 *dies out* if $\xi_t \equiv 0$ for some t. This case is not as easy to handle as Theorem 2, in particular the self-organization time is not uniformly bounded as isolated 1's and 0's may persist for arbitrarily long time: 0001^∞ and 011^∞ are periodic with temporal periods 2 and 1, respectively.

Theorem 3. *Every seed for Quota with $\theta = 2$ is either a maternal replicator or it dies out. The seeds that die out are exactly those that have only isolated 1's separated by at least 3 0's.*

This is one of the rare nontrivial cases for which the entropy for all qualitatively different evolutions is computable. We state the result, the proof of which is a computational exercise, below.

Corollary 4.1. *Assume Quota CA with $\theta = 2$. Large seeds are maternal replicators with probability approaching 1, hence entropy 1. The entropy of seeds that die out equals* $\log_2 \lambda \approx 0.4650$*, where λ is the largest root of $\lambda^4 - \lambda^3 - 1 = 0$.*

Proof. For a configuration ξ_t, let $\iota(\xi_t)$ be the position of its leftmost site whose state is different from that of both neighbors. We will prove in Step 1 below that $\iota(\xi_t)$ is a Lyapunov function for *Quota*, which drives ξ_t into a regular configuration in which all blocks are of length at least 2; the Jen regime then proceeds until all the blocks but possibly one have even length. Thereafter the dynamics is conjugate to *Rule 90*. After Step 1, the argument for the first claim in Theorem 3 is thus very similar to the ones in [**Jen4**], [**GG4**], or Theorem 2 above, so it is omitted.

Step 1 (self-organization). For all t:

$$\max\{\iota(\xi_{t+1}), \iota(\xi_{t+2})\} \geq \iota(\xi_t) + 1.$$

Therefore, all blocks have length at least 2 for large enough t.

This claim involves finitely many configurations and can be easily checked by computer, but we find a written-out proof much more illuminating.

We assume, without loss of generality, that $\iota(\xi_0) = 0$, and that the configuration near the origin (underlined) is $00\underline{1}0$. First we investigate what might happen in a single time step. To see whether ι moves at least one site to the right, we need to consider all possibilities for initial configuration two sites to the right, and three sites to the left, of the $00\underline{1}0$ segment. In Table 1 we label by $*$ the sites whose state does not matter and give below the initial configuration the states at time 1, which are determined from the information given. In square brackets we give a lower bound on $\iota(\xi_1)$.

$(a)\ *\,0000\underline{1}000$ $000\underline{0}0$ [+2]	$(b)\ *\,0000\underline{1}001$ $000\underline{0}11$ [+3]	$(c)\ *\,0000\underline{1}010$ $000\underline{1}11$ [+3]
$(d)\ *\,0000\underline{1}011$ $000\underline{1}1$ [+2]	$(e)\ 11000\underline{1}000$ $1110\underline{0}0$ [+2]	$(f)\ 11000\underline{1}001$ $1110\underline{0}11$ [+3]
$(g)\ 11000\underline{1}010$ $1110\underline{1}1$ [−1]	$(h)\ 11000\underline{1}011$ $1110\underline{1}1$ [−1]	$(i)\ *\,1100\underline{1}000$ $111\underline{0}0$ [+2]
$(j)\ *\,1100\underline{1}001$ $111\underline{0}11$ [0]	$(k)\ *\,1100\underline{1}010$ $111\underline{1}11$ [+3]	$(l)\ *\,1100\underline{1}011$ $111\underline{1}1$ [+2]

Table 1. Possible configurations after one time step.

Now we observe that only cases (g), (h) and (j) need to be considered further. Each of these (by the complement property) leads to one of the cases (a)–(f) at time 1, hence $\iota(\xi_2) \geq 2-1 = 1$.

We now proceed to the proof of the second claim.

Step 2. If ξ_0 contains a 11 pattern, so does ξ_1.

To prove this, consider the *leftmost* 11, and assume that its left site is located at 0. Each of the three possibilities $00\underline{1}1$, $01\underline{1}1$, or $10\underline{1}1$ yields at the next time a 11 to the left of the origin, the first placed at -1, and the remaining two at -2.

To finish the proof of the second claim in Theorem 3, observe first that if there is no 11 in ξ_0, but there are two 1's at distance at most 3, then there is a 11 in ξ_1. The condition given is by Step 2 thus necessary for a seed to die out.

To prove sufficiency assume that every pair of 1's in ξ_0 is at distance 4 or more. Then the same holds for ξ_1 (as such pair can only generate a 1 at the midpoint between them when they are at distance 4), and then, by Step 1, $\xi_t \equiv 0$ for some t. □

5 Replicators in *Perturbed Exactly 1*

We begin by the count of replicators for seeds of length $n + 2$ up to 17. Our method checks for the "mass extinction" signature of replication by a time cutoff: the evolution runs to time $t = 2000$, and once $t > 200$ we check whether the density between the extreme 1's changes from above 0.9 to below 0.1. Maternal cases are then distinguished as those for which the occupied sites after the density drop consist of two configurations that are equal up to translation. The successful pass of this check produced a replicator in every case we investigated further, although we do not have a proof that this method is completely reliable. The results are shown in Table 2 below. We remark that in most cases (i.e., in all chaotic ones) we have no argument that would preclude a later onset of replication, so we can only produce lower bounds.

n	all	maternal	fraternal
0	1	1	0
1	2	2	0
2	4	4	0
3	8	8	0
4	16	8	8
5	32	28	4
6	64	30	34
7	128	103	23
8	249	79	170
9	512	398	114
10	975	296	679
11	2046	1500	538
12	3907	720	3187
13	8156	5941	2215
14	15265	2952	12313
15	13957	10050	3907

Table 2. Replicator counts for small seeds.

It is remarkable that, modulo mirror images, only four out the 1024 smallest seeds, i.e., those of length at most 11, are not replicators. The four "anomalies" have length 10: 1001100111 and 1100101111 both lead to the quasireplicating seed of Theorem 5, 1100110011 is presumably another quasireplicator (see Section 6 for evidence), and 1001001101 appears to be a chaotic seed in the sense of [**GG4**]. The resulting (maximal) chaotic wedge has density 0.375, and other properties similar to *Exactly 1* chaos, except that there seem to be no quasiperiodic regions, due to the fact that the wedge spreads slower than the speed of light.

Prevalence for replicators for smaller seeds has its counterpart in relatively large replicator entropy constants, to which we now turn our attention.

Theorem 4. *The entropy of odd maternal replicators, and that of fraternal replicators, are each at least* 0.8923*. The entropy of even maternal replicators is at least* 0.7741*.*

We begin by three propositions which will establish sufficient conditions for various types of replication. We call an initial state ξ_0 *additive* if each 1-block is of odd length and each 0-block is of odd length at least 3.

Proposition 5.1. *Assume that ξ_0 is additive and placed so that the leftmost occupied site is at the origin. Create the initial state λ_0 for Rule 90 by changing all 1's in ξ_0 at odd locations to 0. Then ξ_t is obtained by changing every isolated 0 in λ_t to 1. In particular, ξ_t is additive for all t, and a maternal replicator.*

Proof. Clearly it is enough to verify the statement for $t = 1$. We proceed two steps.

Step 1. For odd x, $\xi_1(x) = \lambda_1(x)$.

Assume that the ξ_0 states in the range 2 neighborhood of x are $abcde$. Under our conditions, none of a, c, and e can be an endpoint of a block of 1's. Thus by symmetry there are five cases we have to check: 00000, 11000, 11110, 11111, 01110, and 01000. In the second and the last case $\xi_1(x) = \lambda_1(x) = 1$, and in the other three cases they are both 0.

Step 2. For even x, $\xi_1(x) = 1$ if and only if $\lambda_1(x-1) = \lambda_1(x+1) = 1$. (Note that $\lambda_1(x) = 0$.)

Now we need to check all possible ξ_0 configurations in $[x-3, x+3]$. By symmetry, there are nine of them: 0000000, 0100000, 0111000, 0111110, 1100000, 1111000, 1111110, 0100010, 1111111. In order, they result in the following configurations (ξ_1, λ_1) within $[x-1, x+1]$: $(000, 000)$, $(100, 100)$, $(001, 001)$, $(000, 000)$, $(100, 100)$, $(001, 001)$, $(000, 000)$, $(111, 101)$, $(000, 000)$, verifying Step 2. $\square$

Proposition 5.2. *Assume that all 1's in ξ_0 are isolated and the length of all blocks of 0's has length* 3 mod 4*, except that a single block of 0's has even length. Then ξ_0 is an even maternal replicator.*

Proof. We refer back to the previous proposition, and note that in the absence of the irregular block the comparison dynamics λ_t starts from λ_0 which has 1's only on locations 3 mod 4. This is then true for λ_t at every even time t, while at odd times t, $\lambda_t(x) = 1$ implies that at least one of $\lambda_t(x-1)$, $\lambda_t(x+1)$ is also 1. This claim is easily proved by induction.

Consequently, $\xi_t = \lambda_t$ for even t, while for odd t all blocks of 1's in ξ_t are of length at least 3. Chose a block of 0's and track its successors through time: if its length is at least 7, it shrinks by 2 at each of the next two time steps; if its length is 3, its successor is a block of 1's of length at least 7, and then a block of at least 7 0's at the next time.

Remove a 0 from the chosen 0-block. If it is now of length 2, its successor is an even block of 1's of length at least 6, and then a 0-block of length at least 6. If the initially modified block is of length at least 6, then it merely shrinks by 2, and then by 2 again.

Now *add* a 0 to the chosen 0-block. If it is now of length 4, its successor is a block of two 0's flanked by two 1-blocks of length at least 3, resulting next time in an even 0-block of length at least 8. Again, if the initially modified block is longer (now of length at least 8), it merely shrinks by 2 in each of the next time steps.

Therefore, if one starts from one of the assumed initial conditions, and the irregular block is of length 0 mod 4 or 2 mod 4, the resulting dynamics is conjugate to the one with the block "fixed" by, respectively, adding or removing one site. □

We remark that Proposition 5.2 does not hold for a larger number of even blocks, as their interactions may easily destroy the conjugacy.

Next, we construct fraternal replicators by suitable edge modifications.

Proposition 5.3. *A seed η is placed so that its leftmost 1 is at the origin. Assume that another seed η' equals η on nonnegative sites, and that η' is of one of the two types below. Then $\xi_t^{\eta}(x) = \xi_t^{\eta'}(x)$ for $x \geq -t$, $t \geq 0$.*

Thin configuration. Assume that $\eta(0) = \eta(1) = \eta(2) = 1$. In addition, to the left of the origin, η' only has isolated 1's and blocks of 0's of lengths 3 mod 4*, except for the block of 0's immediately to the left of the origin, which has length* 0 mod 4*.*

Thick configuration. Alternatively, assume that $\eta(0) = 1$, $\eta(1) = \eta(2) = 0$. Now require that, to the left of the origin, η' only has blocks of 1's of length 3 mod 4 *and blocks of 0's of lengths* 1 mod 4*, except for the block of 0's immediately to the left of the origin, which has length* 2 mod 4*.*

Proof. Start by observing that the leftmost three states of the configuration ξ_t^{η} are in a cycle 111–100 of length two.

Form η'' by erasing all 1's from η' on nonnegative sites, and run the dynamics for a single time step. As the proof of Proposition 5.1 demonstrates, a configuration of the thick type turns into configuration of the thin type, and vice versa (except for the now meaningless requirement on the block of 0's to the left of the origin).

Thus, we need to verify that the configuration $\xi_1^{\eta'}$ at time 1 in $[-2, 1]$ is 0100 in the thin case and 0111 in the thick case, and that the block of 0's that covers -1 has the correct length mod 4. But the check of this is trivial for the thin type, and also for the thick type when the block of 0's to the left of the origin has length at least 6. When the η' configuration in $[-5, 2]$ is 11100100, then $\xi_1^{\eta'}$ in $[-5, 1]$ is 0000111, and the length of the block of 0's is larger by a single site from what it would be if started from η'', thus of length 2 mod 4. □

We now proceed to prove Theorem 4 in three separate cases.

Transitions from the 12 0-types:

$$\begin{matrix}0&0&0&0\\&&\underline{0}&\end{matrix} \quad \nearrow \begin{matrix}0&0&0&0\\&&\underline{0}&\end{matrix} \quad \searrow \begin{matrix}0&0&0&1\\&&\underline{1}&\end{matrix}$$

$$\begin{matrix}0&0&0&1\\&&\underline{1}&\end{matrix} \quad \nearrow \begin{matrix}0&0&1&0\\&&\underline{1}&\end{matrix} \quad \searrow \begin{matrix}0&0&1&1\\&&q&\end{matrix}$$

$$\begin{matrix}0&1&0&0\\&&\underline{1}&\end{matrix} \quad \nearrow \begin{matrix}1&0&0&0\\&&q&\end{matrix} \quad \searrow \begin{matrix}1&0&0&1\\&&q&\end{matrix}$$

$$\begin{matrix}0&1&0&1\\&&\underline{0}&\end{matrix} \longrightarrow \begin{matrix}1&0&1&1\\&&\underline{0}&\end{matrix}$$

$$\begin{matrix}0&1&0&1\\&&\underline{1}&\end{matrix} \longrightarrow \begin{matrix}1&0&1&0\\&&q&\end{matrix}$$

$$\begin{matrix}1&0&0&0\\&&\underline{0}&\end{matrix} \longrightarrow \begin{matrix}0&0&0&0\\&&\underline{0}&\end{matrix}$$

$$\begin{matrix}1&0&0&0\\&&\underline{1}&\end{matrix} \longrightarrow \begin{matrix}0&0&0&1\\&&\underline{1}&\end{matrix}$$

$$\begin{matrix}1&0&0&1\\&&\underline{0}&\end{matrix} \longrightarrow \begin{matrix}0&0&1&1\\&&q&\end{matrix}$$

$$\begin{matrix}1&0&0&1\\&&\underline{1}&\end{matrix} \longrightarrow \begin{matrix}0&0&1&0\\&&\underline{1}&\end{matrix}$$

$$\begin{matrix}1&1&0&0\\&&\underline{0}&\end{matrix} \longrightarrow \begin{matrix}1&0&0&1\\&&q&\end{matrix}$$

$$\begin{matrix}1&1&0&0\\&&\underline{1}&\end{matrix} \longrightarrow \begin{matrix}1&0&0&0\\&&q&\end{matrix}$$

$$\begin{matrix}1&1&0&1\\&&\underline{0}&\end{matrix} \quad \nearrow \begin{matrix}1&0&1&0\\&&q&\end{matrix} \quad \searrow \begin{matrix}1&0&1&1\\&&\underline{0}&\end{matrix}$$

Transitions from the 11 1-types:

$$\begin{matrix}0&0&1&0\\&&\underline{1}&\end{matrix} \quad \nearrow \begin{matrix}0&1&0&0\\&&\underline{1}&\end{matrix} \quad \searrow \begin{matrix}0&1&0&1\\&&q&\end{matrix}$$

$$\begin{matrix}0&0&1&1\\&&\underline{0}&\end{matrix} \longrightarrow \begin{matrix}0&1&1&1\\&&\underline{0}&\end{matrix}$$

$$\begin{matrix}0&1&1&0\\&&\underline{1}&\end{matrix} \longrightarrow \begin{matrix}0&1&1&0\\&&q&\end{matrix}$$

$$\begin{matrix}0&1&1&0\\&&\underline{0}&\end{matrix} \longrightarrow \begin{matrix}1&1&0&1\\&&\underline{0}&\end{matrix}$$

$$\begin{matrix}0&1&1&0\\&&\underline{1}&\end{matrix} \longrightarrow \begin{matrix}1&1&0&0\\&&q&\end{matrix}$$

$$\begin{matrix}0&1&1&1\\&&\underline{0}&\end{matrix} \quad \nearrow \begin{matrix}1&1&1&0\\&&\underline{0}&\end{matrix} \quad \searrow \begin{matrix}1&1&1&1\\&&0&\end{matrix}$$

$$\begin{matrix}1&0&1&0\\&&\underline{0}&\end{matrix} \longrightarrow \begin{matrix}0&1&0&1\\&&q&\end{matrix}$$

$$\begin{matrix}1&0&1&0\\&&\underline{1}&\end{matrix} \longrightarrow \begin{matrix}0&1&0&1\\&&\underline{1}&\end{matrix}$$

$$\begin{matrix}0&1&1&1\\&&\underline{0}&\end{matrix} \quad \nearrow \begin{matrix}0&1&1&0\\&&q&\end{matrix} \quad \searrow \begin{matrix}0&1&1&1\\&&0&\end{matrix}$$

$$\begin{matrix}1&1&1&0\\&&\underline{0}&\end{matrix} \quad \nearrow \begin{matrix}1&1&0&0\\&&q&\end{matrix} \quad \searrow \begin{matrix}1&1&0&1\\&&\underline{0}&\end{matrix}$$

$$\begin{matrix}1&1&1&1\\&&\underline{0}&\end{matrix} \quad \nearrow \begin{matrix}1&1&1&0\\&&\underline{0}&\end{matrix} \quad \searrow \begin{matrix}1&1&1&1\\&&\underline{0}&\end{matrix}$$

Table 3. Transition rules for the quintuples.

Proof. (*Odd maternal case.*) We suitably modify the method from [**GG4**], by counting predecessors of additive configurations, i.e., configurations that lead to an additive state after j-steps. Essentially, we reinterpret de Bruijn diagram ideas [**Jen1**] in a format suitable for efficient sparse matrix computations. To do this, we use j-*quintuples*, which are defined recursively: a $(j+1)$-quintuple is a scheme $\begin{matrix} a & b & c & d \\ & & e & \end{matrix}$, where $a, b, c, d \in \{0, 1\}$ and e is a j-quintuple. We call c the *type* of quintuple.

We restrict the type of e to be a possible result of the *Perturbed Exactly 1* rule at a site whose neighborhood states are $abcd0$ or $abcd1$. Therefore, the constraint on the type of e is as follows: If both $abcd0$ or $abcd1$ produce the same state s, then the quintuple e must be of type s. For $j \geq 0$, consider two $(j+1)$-quintuples with identical first rows and their two e's of the same type *similar*. We have 23 possible similarity classes, 12 0-types and 11 1-types, listed on the left sides of the transitions in Table 3, where the underlined 0's and 1's represent arbitrary j-quintuples of respective types.

The recursive construction starts with 0-quintuples. By definition, there are five of them, ordered as given here: three 0-types, which we denote by 0_1, 0_o, 0_e, and two 1-types, which we denote as 1_o, and 1_e. These are used to count additive configurations. Namely, we need to enumerate the number of paths from 1_o to 1_o in a directed graph with connections $1_o \to 0_1$, $1_o \to 1_e$, $1_e \to 1_o$, $0_1 \to 0_e$, $0_e \to 0_o$, $0_o \to 0_e$, $0_o \to 1_o$. (Here, the e and o subscripts stand for elements that are at even and odd locations within a particular block, and the 0_1 stands for the first zero after a block of ones.)

The j-step predecessors will be represented as paths through j-quintuples, thus we recursively list transitions for $(j+1)$-quintuples in Table 3. An arbitrary j-quintuple of type 0 is represented by $\underline{0}$, and $\underline{1}$ has analogous meaning; these are possibly different on the left and right side of any transition. Moreover, q represents an arbitrary j-quintuple, and transitions are thus determined recursively by the possible j-quintuple transitions (satisfying the restrictions on type).

For example, a $(j+1)$-quintuple similarity class $Q_5 = \begin{matrix} 0 & 1 & 0 & 1 \\ & & \underline{1} & \end{matrix}$ implies that a site x is in state 1 at some time t, and its neighborhood state is $0101s$ at time $t-1$, for some $s \in \{0, 1\}$. But then the CA rule dictates that s is 0, and so the first four states of the neighborhood of $x+1$ at time $t-1$ are 1010. As neighborhood configurations 10100 and 10101 produce different results, there is no restriction on the type of the j-quintuple on the bottom row; therefore, in Table 3, Q_5 transitions to $\begin{matrix} 1 & 0 & 1 & 0 \\ & & q & \end{matrix}$. The interpretation is that all possible transition from a j-quintuple of type 1 to a j-quintuple of any type determine all transitions from any representative of Q_5.

The rules in Table 3 translate, via a block substitution rule, into a recursive construction of the transition matrices M, with the base ($j = 0$) matrix given by the transitions for 0-quintuples. For any j, matrices are of the dimensions $n_0 + n_1$, where n_0 and n_1 are the numbers of j-quintuples of the corresponding type. We will not give further details, which proceed along the same lines as in the proof of Theorem 2 in [**GG4**].

This algorithm results in the estimates in Table 4, computed by MATLAB's sparse matrix routines (see the script at [**GGM**], with $\lambda_{\max}$ the maximum eigenvalue of M and the last column rounded down. The final number is featured in the statement of the theorem.

j	$n_0 + n_1$	$\lambda_{\max}$	$\log_2 \lambda_{\max}$
0	5	$\sqrt{2}$	0.5
1	59	1.7578	0.8137
2	680	1.8201	0.8640
3	7862	1.8403	0.8799
4	90860	1.8562	0.8923

Table 4. Entropy estimates for odd maternal replicators based on j-quintuples.

The matrices M are periodic with period 2, so they have eigenvalues $\pm\lambda_{\max}$. Thus it is crucial to note that the path between extreme 1_o's takes an even number of steps. □

Proof. (*Even maternal case.*) We now use Proposition 5.2. The crucial observation is that we can merely count the predecessors of configurations in the statement of that proposition *without* the even block. Indeed, to each such predecessor, a 1 could be added to, say, its right side, separated from the extreme 1 by a sufficiently large even block of 0's. See Fig. 8 for an example.

The method from now on is the same as the previous proof, except for the new definition of 0-quintuples. Again, there are five of them: four 0-types denoted by 0_0, 0_1, 0_2, 0_3, and a single 1-type, denoted simply by 1. The transitions are $0_0 \to 0_1 \to 0_2 \to 0_3 \to 0_0$, and $1 \to 0_1$, $0_3 \to 1$. The resulting estimates are given in Table 5.

j	$n_0 + n_1$	$\lambda_{\max}$	$\log_2 \lambda_{\max}$
0	5	$2^{1/4}$	0.25
1	62	1.6441	0.7172
2	710	1.7017	0.7669
3	8216	1.6806	0.7481
4	94940	1.7101	0.7741

Table 5. Entropy estimates for even maternal replicators based on j-quintuples. □

Proof. (*Fraternal case.*) This is an easy consequence of Proposition 5.3. Namely, observe that both thin and thick types add odd length to a seed. To produce an even fraternal replicator, we can enlarge any seed used in the proof for the odd maternal replicator case by 111 placed on the right after a sufficiently large even block of 0's (of length 2 or 4 mod 2 so that it is in the correct phase when it interacts with the original seed). To make the fraternal replicator odd, we in addition append a thin type, say, 00001. Again, see Fig. 8 for an example. □

Proposition 5.3 also yields a curious property of *Perturbed Exactly 1*: entropy of any non-trivial property that does not depend on the behavior on the edge of the light cone is uniformly bounded from below. Note that replication and quasireplication satisfy the assumption, but maternal replication does not. Due to the example presented in the next section, we thus get a lower bound on the entropy of quasireplicators that are not replicators.

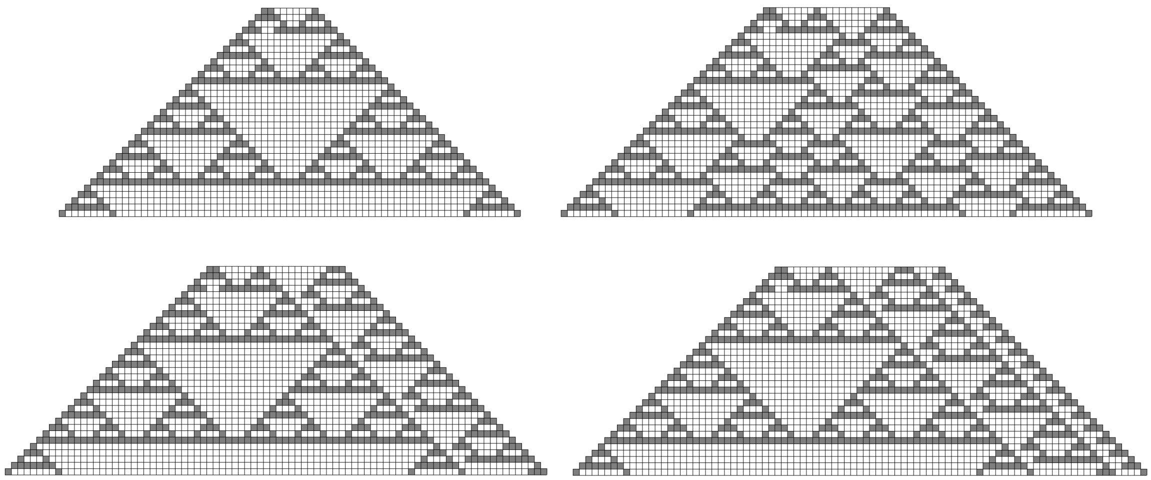

Fig. 8. Constructions in the proof of Theorem 4. Top left is the evolution from seed $A = 110000001$ whose configuration becomes additive in four steps and is therefore an odd maternal replicator. Top right seed, A[10 0's]1, is designed to lead to a configuration from Proposition 5.2 in 4 steps, thus is an even maternal replicator. Bottom two seeds are A[10 0's]111 and A[10 0's]11100001, an even and an odd fraternal replicator.

Proposition 5.4. *If there exist a seed with property $\mathcal{P}$, i.e., $\mathcal{P} \neq \emptyset$, and that, for an arbitrary $k \in \mathbb{Z}$ and any seed, $\mathcal{P}$ only depends on the configuration on $\{(x,t) : x \geq k - t\}$. Then the entropy of $\mathcal{P}$ is at least* 0.8075.

Proof. It is easy to verify that, started from any seed, the left edge of any configuration is either 100 or 111 by time 2: if the left edge is 1010 or 1100 then this already happens at time 1, in the remaining two cases, 1011 and 1101, the left edge is 1100 at time 1.

Thus, by Proposition 5.3, the entropy is at least the entropy of the seeds which start with a single 1, then to the left a configuration of the second type, then to its left a configuration of the first type, etc., and also at least the entropy of their predecessor set.

The base ($j = 0$) directed graph now has 13 vertices, labeled 1 and 0_i^{thin}, 0_i^{thick}, 1_i^{thick}, $i = 0, 1, 2, 3$. Any symbol with subscript i transitions to the same symbol with subscript $(i + 1) \bmod 4$ and in addition we have transitions $1 \to 0_1^{\text{thin}}$, $1_3^{\text{thick}} \to 0_1^{\text{thick}}$, $0_3^{\text{thin}} \to 1$, $0_2^{\text{thin}} \to 1_1^{\text{thick}}$, $0_1^{\text{thick}} \to 1_1^{\text{thick}}$, $0_0^{\text{thick}} \to 1$. The resulting matrix substitution scheme yields the estimates in Table 6.

j	$n_0 + n_1$	$\lambda_{\max}$	$\log_2 \lambda_{\max}$
0	13	1.3339	0.4157
1	154	1.6828	0.7508
2	1774	1.7364	0.7961
3	20512	1.7431	0.8017
4	237052	1.7502	0.8075

Table 6. Entropy estimates for edge modifications based on j-quintuples. □

6 Quasireplicators in *Perturbed Exactly 1*

We begin with the main task of this section, which is to give a complete proof that a particular initial state leads to a fractal Willson limit, but not to replication.

Theorem 5. *Initial state* 1000111[10 0's]1000111 *is a quasireplicator, but not a replicator, for Perturbed Exactly 1.*

Proof. Let $B = 000111$ and consider the seven quasireplicating elements 0; three *L-types*: $1B$, $0B$, and 1; and three *R-types*: $01B$, $00B$, and 01. Recall that the placement of a configuration is the position of its left endpoint.

Step 1. Assume that at time 0 the elements are placed at sites of $16\mathbb{Z}$ in such a way that an L-type is never followed by an R-type if the corresponding entry in Table 7 is empty. The resulting configuration at time 8 consists of elements placed at $16\mathbb{Z} + 8$, each computed by the rule in Table 7: the element at $16k + 8$ at time 8 is obtained as the result of (noncommutative) interaction between its left element at $16k$ and the right element at $16(k + 1)$ at time 0.

Note first that all elements consist of at most 8 sites. Next observe first that the speed of light of *Perturbed Exactly 1* (two spatial units per time unit) implies that the configuration in $[8, 15]$ at time 8 only depends on the initial configuration in $[-8, 31]$. The proof of Step 1 then reduces to finite checking, best done by computer; merely running the dynamics from the assumed initial state, which consists of suitably placed $1B$ and $01B$, one can catalog all interactions by time 128 (and most by $t = 64$, see Fig. 9).

We will now consider the CA with the 7 elements as states, and on space-time sites (x, t) with $x + t$ even. For notational convenience, we will start this CA at time $t = 1$, with $1B$ and $01B$ states at 1 and -1, respectively, and 0's elsewhere. We call the resulting evolution η_t, following rules from Table 7, the B-dynamics. We say that the evolution is *well-defined* up to time T, if the missing pairs in Table 7 do not occur in η_t, $t < T$. We will establish later that B-evolution is well-defined for all time. For a B-configuration η, we define configurations $C(\eta)$, $D(\eta)$, and $N(\eta)$ with states 0 and 1: $C(\eta)(x) = 0$ exactly when $\eta(x) \in \{0, 1, 01\}$, $D(\eta)(x) = 0$ exactly when $\eta(x) \in \{0, 1B, 01B\}$, while $N(\eta)(x) = 0$ exactly when $\eta(x) \in \{0, 0B, 00B\}$.

	0	$1B$	$0B$	1	$01B$	$00B$	01
0	0	$1B$	$0B$	1	$01B$	$00B$	01
$1B$	$1B$	0	1	$0B$	$00B$	1	
$0B$	$0B$	1	0	$1B$		0	
1	1	$0B$	$1B$	0			0
$01B$	$01B$				0	01	$00B$
$00B$	$00B$				01	0	$01B$
01	01				$00B$	$01B$	0

Table 7. Interactions between quasireplicating elements. Left elements are listed by row, and right elements by column.

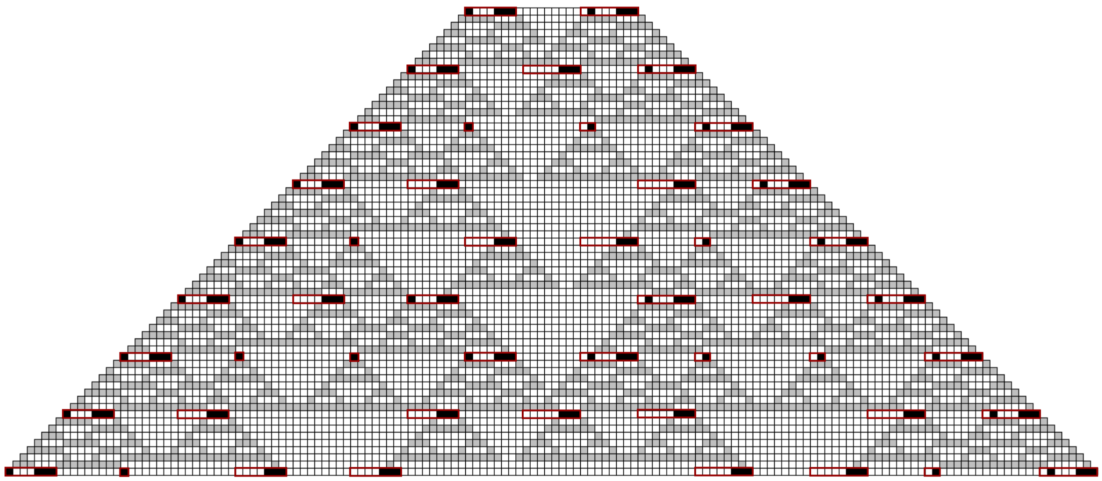

Fig. 9. Evolution of *Perturbed Exactly 1* from the initial state in Theorem 5 up to time 64, with darker configurations at times that are multiples of 8, and highlighted replicating elements
.

Step 2. Assume that the B-dynamics is well-defined up to time T. Then all L-types are to the left of all R-types. Moreover, for $t \leq T$, $N(\eta_t)$ is obtained from $N(\eta_{t-1})$ by *Rule 90*.

The proof is an inspection of Table 7.

Call a B-configuration η_t *symmetric* if the state of $x = 0$ (for even t) is 0, nonzero states occurs at symmetrically placed positions $-x$, x, $x > 0$, and the R-type at x is obtained from the L-type at $-x$ by prefixing a 0.

Step 3. Assume that the dynamics is well-defined up to time t_1. Assume also that the dynamics is symmetric at time t_1 and that for times $t \in [t_1, t_2]$ there is no ordered pair $1B, 01B$ in η_t. Then the configuration is symmetric for $t \in [t_1, t_2 + 1]$, and well-defined up to time $t_2 + 2$. Finally, for $t \in [t_1 + 1, t_2 + 1]$, $C(\eta_t)$ and $D(\eta_t)$ are obtained, respectively, from $C(\eta_{t-1})$ and $D(\eta_{t-1})$ by *Rule 90*.

The proof is another inspection of Table 7, and an easy induction argument for symmetry (note that under symmetry all L-R interactions other than $1B$-$01B$ produce a 0).

The assumptions at Step 3 are satisfied for $[t_1, t_2]$=$[3, 6]$, $[9, 30]$, $[33, 126]$, etc. At time $t = 7$, the pair $1B$, $01B$ appears, which at time 8 produces $00B$ at the origin, which violates symmetry but is annihilated at the next time by interaction between $0B$ to the left and $00B$ to the right. We will show this happens exactly at times 2^n, n odd. Before we do that, we need to study another auxiliary dynamics.

Consider the *Rule 90* CA, started from 1 placed at the origin. By symmetry, the origin is not occupied at any later time. Add an occupied site at the origin at time 2 and then restart the *Rule 90* dynamics. Then make the origin (which would otherwise be empty) occupied at time $2^3 = 8$, then again at time 2^5, and in general at any time 2^n, $n \geq 1$ odd. Call this rule the *C-dynamics*. Once proved, the previous paragraph will imply that 1's in $C(\eta(t))$ are exactly at positions of 1's in the C-dynamics.

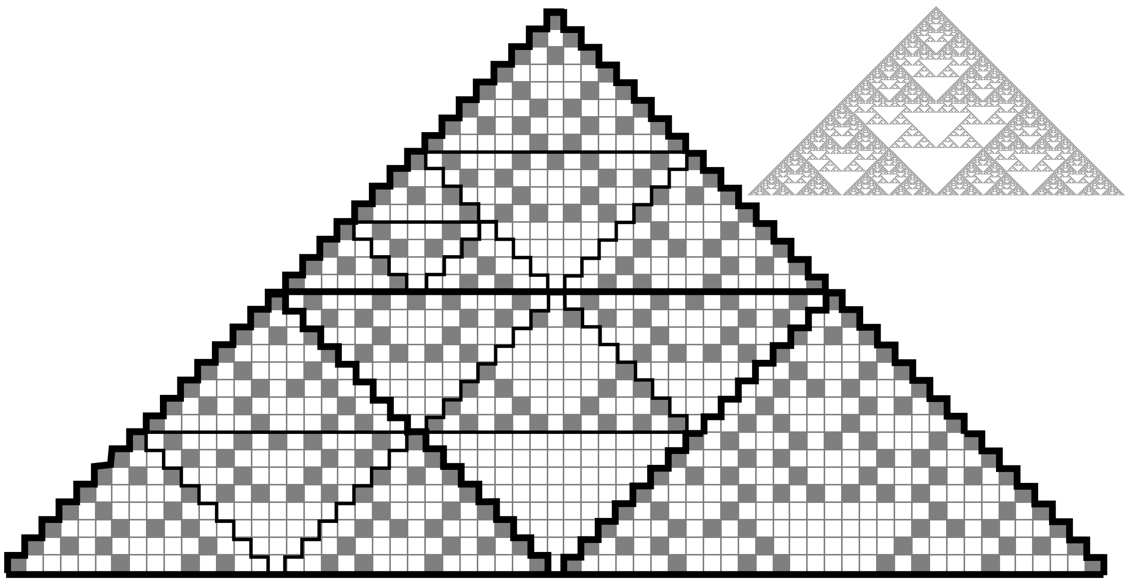

Fig. 10. Evolution of C-dynamics up to time 31, and time 512 (small triangle). The resulting α_5 is divided into α_4 (top), β_4 (bottom left), β_4^R (bottom right), and σ_4 (middle) by the thicker line. Then α_4 is subdivided, and so are β_4, and σ_4; the bottom left subregion of α_4 is β_3, which is further subdivided.

We will consider C-configuration α_n, at time $2^n - 1$ in the space-time triangle $T_n = \{(x,t) : 0 \le t \le 2^n - 1, |x| \le t\}$. We will recursively describe α_n by configurations in T_n or in one of its translates, or in a translate of $T'_n = \{(x,t) : 0 \le t \le 2^n - 1, |x| \le 2^n - 1 - t\}$, the upside-down reflection of T_n. We will use the coordinates in T_n and T'_n, i.e., translate the configurations back to the original position of T_n or T'_n. Observe that α_n is symmetric around the line $x = 0$.

Step 4. The recursions are given in Fig. 11.

Note that the dynamics from time 2^n to time 2^{n+1}, apart from possible addition of the 1 at the origin, is additive. Therefore the C-configurations, at time 2^{n+1}, in $[-2^{n+1}+1, -1]$ and in $[1, 2^{n+1}-1]$ are suitable translations of the one in $[-2^n+1, 2^n-1]$ at time 2^n. By induction, the following statement follows, which we record in a separate Step, for later use.

Step 5. The C-configuration at time 2^n has 1, 1 at positions 2^n, $2^n - 2$; moreover, it has 1, 1 at positions ± 2, while at 0 it has, by the C-rule, 1 or 0 depending on whether n is odd or even.

The recursion for β_n then simply follows from the speed of light, and so does the recursion for σ_n. The recursion for λ_n is an easy exercise (and well-known).

Inductively, σ_n has no 1 on the boundary of T'_n ($|x| = t$) and at the next closest sites ($|x| = t - 1$) only when $t = 2^k - 1$, $k < n$. These generate *Rule 90* configurations unless they get annihilated by the configuration from the previous "nucleus." The final nuclei at $t = 2^{n-1} - 1$ generate a row of 1's 2^{n-2} time units later (including the two nuclei next to the boundary) and a row of 0's at the next time. This establishes the recursion for ν_n.

To establish the recursion for γ_n and δ_n, we argue by additivity that $\gamma_n = \beta_n^R + \lambda_n$ mod 2 for odd n, and thus $\delta_n = (\beta_n^R + \lambda_n)$ mod 2 for even n. For example, the bottom right corner configuration of δ_n is, modulo 2,

$$(\gamma_{n-1} + \lambda_{n-1})^R = \gamma_{n-1}^R + \lambda_{n-1} = \beta_{n-1} + \lambda_{n-1} + \lambda_{n-1} = \beta_{n-1}.$$

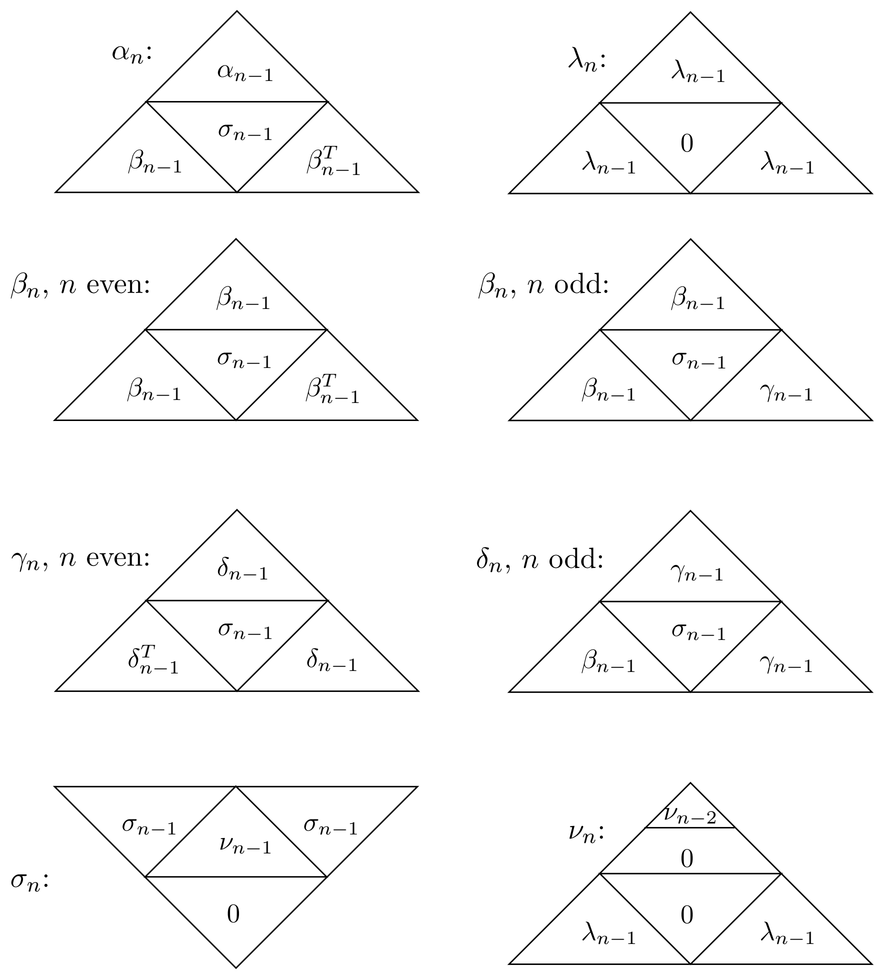

Fig. 11. Recursive specification of α_n. The superscript T indicates the mirror image of a configuration. This implies that the set of space-time occupied sites is exactly solvable [**GG1**]. Here, λ_n is generated by *Rule 90* from a singleton at the top site of T_n. See Fig. 10 for examples.

Call a pair of binary configuration α and α' in T_n *r-close* if for each 1 in α there is a 1 in α' at ℓ^∞ distance at most r, and vice versa.

Step 6. When defined at the same n, each pair of configurations β_n, β_n^R γ_n, δ_n are 2-close.

This easily follows from the Step 4 recursion, by induction.

Step 7. At $x = t$, $t = 0, 1, \dots 2^n - 1$, β_n has states 10[2 1's][4 0's] $\dots$, ending with an interval of 2^{n-1} 0's or 2^{n-1} 1's, depending on whether n is odd or even.

This again follows by induction: the leftmost 1 on the bottom line of λ_{n-1} either annihilates with the rightmost 1 in the bottom line of β_{n-1} (odd n), or generates a 1 which propagates leftward due to the empty bottom triangle of σ_n.

Step 8. The B-dynamics η_t is well-defined for all times. At any time $t \geq 1$, $C(\eta_t)$ is given by the C-dynamics.

Assume inductively that this holds up to time 2^n, for some odd n. By Step 2, up to that time, all states corresponding to 1's in the translations of configuration σ_n are $0B$ or $00B$. Moreover, all 1's in position described in Step 7 must be occupied by $1B$ or $01B$ in the B-dynamics: certainly the top 1 is, and the rest follows by the B-interactions. Thus, by recursion for α_n, B-dynamics produces a $00B$ at $x = 0$ at time 2^{n+1}, which by Step 5 gets immediately annihilated against its occupied neighbors, as the extra 1 does in the C-dynamics. By Step 3 the claim in Step 8 holds up to time 2^{n+2}.

The configuration α_n misses the states 1 and 01 in the B-dynamics, and the next step explains how those are added. The union, $\cup$, of two configurations is simply the `or` operation between them, i.e., it has a 1 exactly where at least one of them has a 1.

Step 9. The nonzero states within T_n in the B-dynamics are located precisely at nonzero positions of $\alpha_n \cup \lambda_n$.

To prove this, we observe that $D(\eta_t)$ evolves by the C-dynamics, except that the first occupied site occurs at $x = 0$ at time 2 is at time 2. Therefore, the positions of nonzero B-states within T_n are at

$$\bigcup_{t \leq n} (C(\eta_t) \cup D(\eta_t)) = \alpha_n \cup (\alpha_n + \lambda_n) = \alpha_n \cup \lambda_n,$$

where the sum is as usual reduced modulo 2.

Step 10. Conclusion of the proof.

We will show that the C-dynamics is not a replicator and that its Willson limit, which is the limit of configurations $2^{-n}\alpha_n$, is fractal. As we will see this limit has Hausdorff dimension the same as that of the limit of $2^{-n}\lambda_n$, which is $\log 3/\log 2$, and then Step 10 finishes the proof.

By Step 6, the odd and even n give the same limit. This limit is a Mauldin-Williams fractal [**MW**, **BM**] and thus has its Hausdorff dimension equal to its box dimension, and determined by the largest eigenvalue of an appropriate matrix given by the recursion in Step 4. The dimension in fact does equal to $\log 3/\log 2$, so it is the same as for the Sierpinski gasket. However, the Hausdorff measure for this exponent is infinite, which is easily shown by the methods in [**MW**]. (The small triangle in Fig. 10 approximates this fractal.)

To demonstrate that the C-dynamics is not a replicator, observe first that the ether could only be 0, and that by induction every line on the top half of the configuration σ_n, $n \geq 2$, contains at least two occupied points. Thus, by iterating the recursion for σ_n, every one of first 2^k, $k \geq 2$ lines contains at least 2^{n-k} points. It follows that the number of occupied points at times $2^n + 2^k$, $n = 1, 2\ldots$, is not bounded for any fixed k, which violates a necessary condition for a replicator.

□

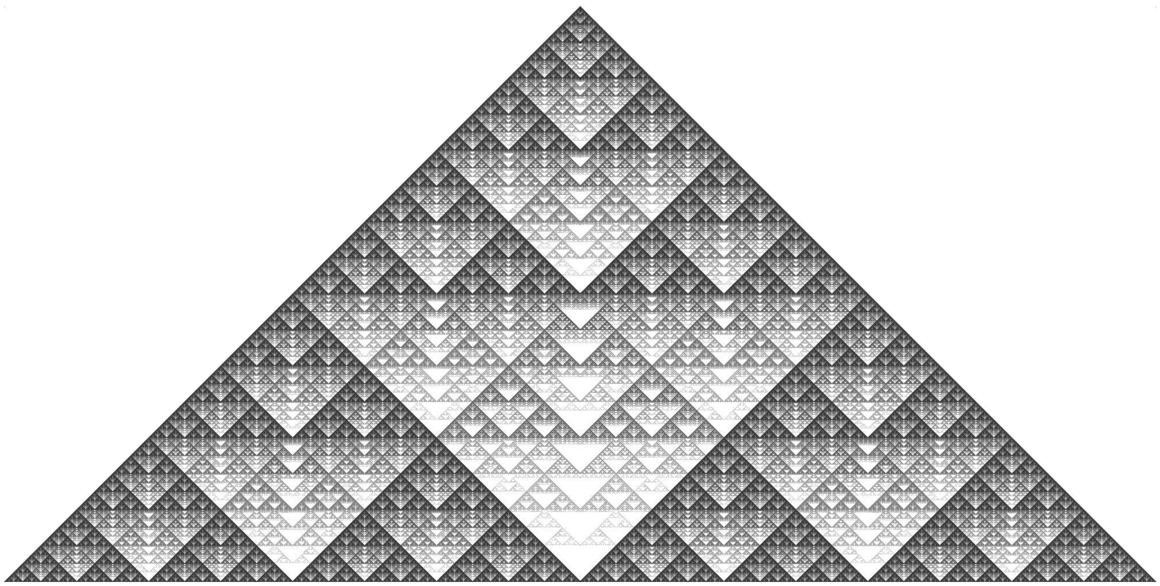

Fig. 12. Complex quasireplication in *Perturbed Exactly 1*.

Perturbed Exactly 1 is apparently capable of another, much more elaborate kind of quasireplication. In Fig. 12, evolution from the seed 1100110011 is pictured at time a little past a million. (To be more precise, the density in 1000×1000 boxes is shown in shades of gray.) The clear message is that of a fractal with the Sierpinski gasket dimension, but also that of an uncertain prospect for a recursive description and a significant challenge for rigorous analysis.

7 Replication and quasireplication in *Extended 1 Or 3*

For a configuration $\eta \in \{0, 1, 2, 3\}^{\mathbb{Z}}$, we denote by η mod 2 the configuration obtained by reducing every state of η modulo 2. We call the seeds ξ_0 such that ξ_0 mod $2 \neq 0$ *genuine*; to avoid trivial cases we will consider only these in this section. As we will see, replication is very common among genuine seeds. The reasons for this are deserving of a thorough investigation, which we will not attempt here. Instead, we provide a couple of modest initial results, the first of which is a sufficient condition on maternal replicators.

We begin with two important observations. First, it is a straightforward to verify that ξ_t mod 2 is the *1 Or 3* CA (as it must be because it represents the "first level" *Box 13* EBD).

Moreover, if the initial state consists only of 0's and 2's (i.e., there are no first level sites), then $\xi_t/2$ is again *1 Or 3*.

Theorem 6. *Assume that a seed has only 0's and 3's. Mark a site if it is either in state 3 with both neighbors in state 3 or both neighbors in state 0, or in state 0 with a neighbor in state 3 and the other in state 0. Assume the distances between successive marked sites are all even. Then the seed is a maternal replicator. In particular, the entropy of odd genuine maternal replicators is at least* 0.25*.*

Proof.

Step 1. Assume that $\xi_0 \in \{0,3\}^{\mathbb{Z}}$ and that no marked sites are neighbors at time 0 (but can be otherwise at an odd distance). Then $\xi_2 \in \{0,3\}^{\mathbb{Z}}$.

To verify this, we pick an x and consider all possible ξ_0 configurations in $[x-2, x+2]$. Up to symmetry, there are 20 of them. Of these, six (00030, 00303, 03003, 03033, 30033, 33333) have marked neighboring sites in $[x-1, x+1]$. Further six (00000, 00333, 03030, 30003, 03303, 33033) result in $\xi_2(x) = 0$, and the remaining eight in $\xi_2(x) = 3$.

Step 2. If the stated assumptions hold for ξ_0, they do so for ξ_2.

Observe that, at time 0, x is marked exactly when $\xi_1(x) \bmod 2 = 1$. Therefore, by Step 1, we need to show that if *1 or 3*, that is, $\lambda_t = \xi_t \bmod 2$, has only even sites occupied in λ_0, the same is true for λ_2. By cancellative duality [**Gri**], for any y,

$$\lambda_2(y) = |\lambda_2^{\{y\}} \cap \lambda_0| \bmod 2 = |\{y-2, y, y+2\} \cap \lambda_0| \bmod 2,$$

and the intersection is clearly empty if y is odd.

To finish the proof, merely note that λ_t replicates and at times $t = 2^n$, $n \geq 1$, $\xi_t = 3 \cdot \lambda_t$. The entropy statement follows from the fact that building a configuration with stated properties from the leftmost 3 rightward one has two choices at each odd step and a single choice (albeit dependent on the preceding choices) at each even step. □

Call a replicator for *Extended 1 Or 3 regular* if, perhaps after an enlargement of n_0, there exist a replicating element A such that either the interaction $A \oplus 0_\pi \oplus 0_\pi$ occurs and equals A, or the same holds for the interaction $0_\pi \oplus 0_\pi \oplus A$. In other words, spreading into the ether, A creates an appropriately positioned copy of itself in some 2^i time steps. To date, every replicator we have checked turned out to be regular. The following proposition exploits the connections to the "ordinary" *1 Or 3* rule.

Proposition 7.1. *Every ether for an Extended 1 Or 3 replicator consists only of 0's and 2's. After all 2's are replaced by 1's, an ether is a periodic solution for 1 Or 3.*

Assume ξ_0 is a regular replicator. Then σ is a power of 2; moreover, if $\sigma \geq 2$, $\tau = \sigma/2$.

Proof. Again, denote $\lambda_t = \xi_t \bmod 2$. For any seed, *1 Or 3* replicates with 0 ether. To be more precise, let A_k be the occupied set of $\lambda_k^{\{0\}}$. Then there are times t_1 and 2^{n_1} so that, for all $k = 1, 2, \ldots$, the configuration of λ_t at times $t_k = t_1 + (k-1)2^{n_1}$ consists of identical finite configurations placed at positions of $2^{n_1}A_k$, with 0's elsewhere. Therefore, at times t_k+1, the *Extended 1 Or 3* dynamics creates only 2's outside of a uniformly bounded neighborhood of $2^{n_1}A_k$. If replaced by 1's, these 2's obey the *1 Or 3* rule. This proves assertions in the first paragraph.

For a regular replicator, there must exist an i so that the ether π agrees with its translation by 2^i, hence σ must divide 2^i. For the last claim, we need to show that any periodic solution for *1 Or 3*, whose spatial period $\sigma \geq 2$ is a power of 2, has temporal period $\tau = \sigma/2$. Starting from a single 1 at the origin, *1 Or 3* generates, at time $\sigma/2$, 1's at 0 and at $\pm\sigma/2$. Therefore, by additivity, if one starts with 1's at $j\sigma$, $j \in \mathbb{Z}$, this state is reproduced at time $\sigma/2$. Another application of additivity demonstrates that τ divides $\sigma/2$ for any state with spatial period σ. To finish the proof, we need to show that τ does not divide $\sigma/4$.

As the spatial period is σ but not $\sigma/2$, we can assume, via a translation, that for all $j \in \mathbb{Z}$, $2j\sigma/2$ contain 1's and $(2j-1)\sigma/2$ contain 0's. If the 1 at the origin is to be reproduced at time $\sigma/4$, either $\pm\sigma/4$ both contain 0's, or both contain 1's. But in either case the two states are not reproduced at time $\sigma/4$. □

We now proceed with an empirical study of occurrence of different ethers. We will not distinguish an ether from its spatial translation, from any of its iterates under the *Extended 1 Or 3* rule, or their mirror images. The evolution of a periodic solution for *1 Or 3* with given σ and τ is given by a configuration on a discrete torus with rows indexed by $0, \dots, \tau-1$ and columns by $0, \dots, \sigma-1$. Rotations give $\sigma \cdot \tau$ possible first rows, and the reflection multiplies this number by 2. (They are not necessarily all different.) Interpret the first row as a binary representation of a number and chose the one with the *smallest* such number among all possibilities. This gives (via the above proposition) the *signature* of an *Extended 1 Or 3* ether. For example, the signature of the only ether we found with $\sigma = 4$ is 0001 and thus it can be given by 0002^∞.

no.	τ	σ	ρ	freq.	signature	ξ_0
1	1	1	0	0.654	0	1
2	1	2	1/2	0.064	01	303
3	2	4	1/2	0.029	0001	1001001
4	4	8	3/8	0.092	[7]1	10111
5	4	8	3/8	0.009	[5]101	100110011
6	8	16	5/16	0.006	[15]1	1000001001
7	8	16	5/16	0.003	[11]10001	300000003
8	8	16	3/8	0.013	[9]1000101	10010101
9	8	16	7/16	0.009	[7]101010101	1000011001
10	32	64	97/256	0.003	[11]1[9]100010101[9]1[7]1[5]10101000101	1101000111

Table 8. Ethers emerging from seeds of length at most 10. The first column is the serial number, the fourth column the frequency among genuine seeds of length at most 10, and the last column the simplest generating seed. In the signatures, $[k]$ stands for an interval of k 0's. Other information is self-explanatory. The total proportion of all replicators is 0.883.

We look for replicators via the following algorithm. Start by picking two positive integers d and $\sigma_{\max}$. Build two sequences of length 2^{d-1} at time $t = 2^d + 2^{d-3}$: $(a_0, \ldots, a_{2^{d-1}-1}) = (\xi_t(x) : -2^d + 2^{d-2} < x \leq -2^{d-2})$ and $(b_0, \ldots, b_{2^{d-1}-1}) = (\xi_t(x) : 2^{d-2} \leq x < 2^d - 2^{d-2})$. Then check whether $a_i = a_{i+\sigma} = b_{i+p}$ for some $0 \leq p < \sigma \leq \sigma_{\max}$ and all $i < 2^{d-1} - \sigma$. As in Section 5, a successful pass of this check does not constitute a proof that a seed is a replicator, but again it appears to be a reliable sufficient condition for the choices we use, $d = 12$ and $\sigma_{\max} = 128$. First we ran this algorithm on every seed of length at most 10 and found 10 different ethers, which are given in the Table 8.

It is not feasible to analyze all seeds of larger lengths, so we resorted to random samples, and found that the proportion of replicators rapidly approaches 1. In fact, in a sample of 10^5 randomly chosen seeds of length at most 100, every single instance was a replicator with $\sigma \leq 64$. There were 152 different ethers, and the most common were nos. 1, 4, 2, 3, 5, and 9, with respective frequencies 0.64, 0.1, 0.05, 0.05, 0.03, and 0.02.

How confident can we be that not all seeds are replicators? Certainly there are mixed examples, but we are more interested in genuinely different behaviors. For example, the seeds 110111 and 1000011 appear to exhibit (different) chaotic dynamics during the first few thousand time steps, but then nearly (but not quite) replicate in the next million steps. It is at least clear that each of these two has a very long self-organizing epoch, and consequently our computations fail to suggest a coherent hypothesis.

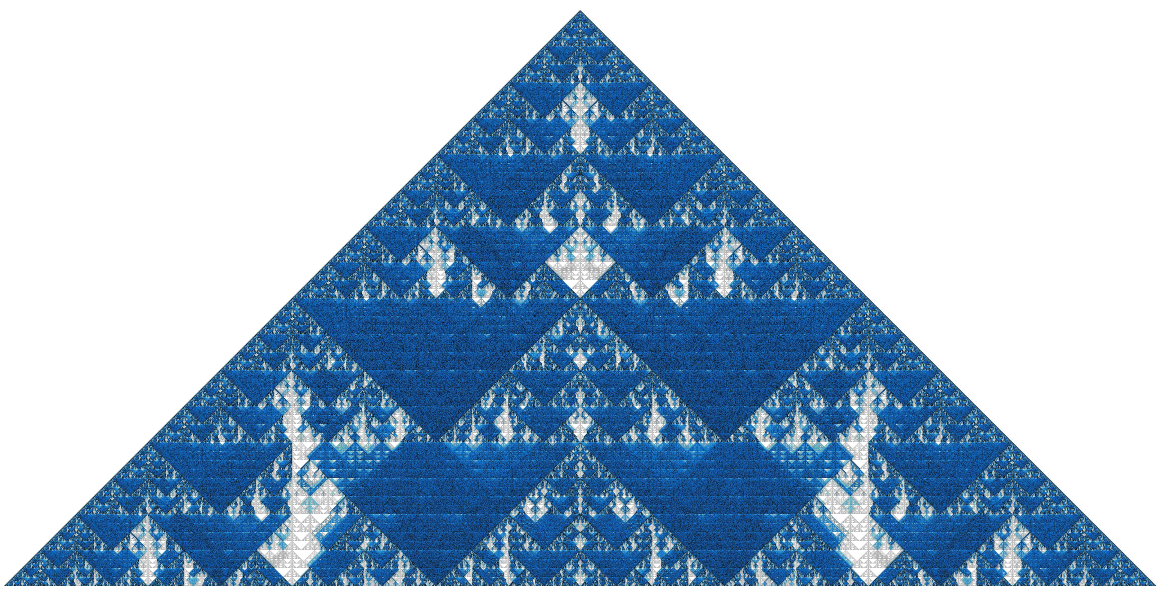

Fig. 13. *Extended 1 Or 3* from $\xi_0 = 3$ at time 2^{20}. Densities in $2^9 \times 2^9$ boxes are represented in progressively darker colors.

By far, the most intriguing case is presented by the simple seed 3 (and many others, such as 111 or 30303, that behave similarly). The space-time configuration at time 2^{20} is depicted in Fig. 13; while later replication appears very unlikely, we do not hazard a more precise conjecture about asymptotic properties due to the extraordinary complexity of the pattern.

Finally, we do have an example which is provably non-replicating and a sketch of its analysis is a fitting conclusion to the paper.

Theorem 7. *Initial state 30003 is a quasireplicator, but not a replicator, for Extended 1 Or 3.*

Proof. We present the main steps omitting most of the tedious checking. The key to the analysis clearly are the more involved dynamics of 2's within some of the inverted triangles left empty by ξ_t mod 2 (see Fig. 14); we call these the *relevant* triangles. All the facts that need to be established are about the *1 Or 3* dynamics λ_t, which will be the default CA for the rest of the proof.

The dynamics of 2's within a relevant triangle is determined by the distribution of single 2's along its top. These nuclei are spaced at intervals that increase as powers of 2, and then result in a *single* nucleus in the triangle below on the same size, which in turn generate the next nucleus on the relevant triangle of twice the size. Our first step implies that this pattern persists on all scales.

Step 1. Fix an $n \geq 3$ and assume that $\lambda_0 = A$, where A is restricted to $[-2^n, 2^n]$ with 1's at $2^n - 12 \cdot 2^k$, $k = 0, 1, \ldots, n-3$. Then, at time $t = 2^n - 3$, $\lambda_t(0) = 1$.

By cancellative duality [**Gri**], we need to show that at the specified t, $\lambda_t^{\{0\}} \cap A$ has odd cardinality. Now, as is easy to prove by induction, $\lambda_t^{\{0\}}$ consists of intervals of three 1's centered at $12i$, $i = 0, \pm 1, \pm 2, \ldots$, $12i + 4$, $i = 0, 1, 2, \ldots$, and $12i - 4$, $i = 0, -1, -2, \ldots$. Further, 2^n mod 12 is either -4 or 4, when $n \geq 3$ is odd or even, respectively. Therefore, for odd $n \geq 3$, $\lambda_t^{\{0\}} \cap A$ consists of a single 1 at $2^n - 12 \cdot 2^{n-3} = -2^{n-1}$. For even $n \geq 3$, the intersection consists of the positive locations of 1's in A, and there is $n - 3$ of them, an odd number.

Next step establishes that the 2's do extend their influence only through the apex of a relevant inverted triangle.

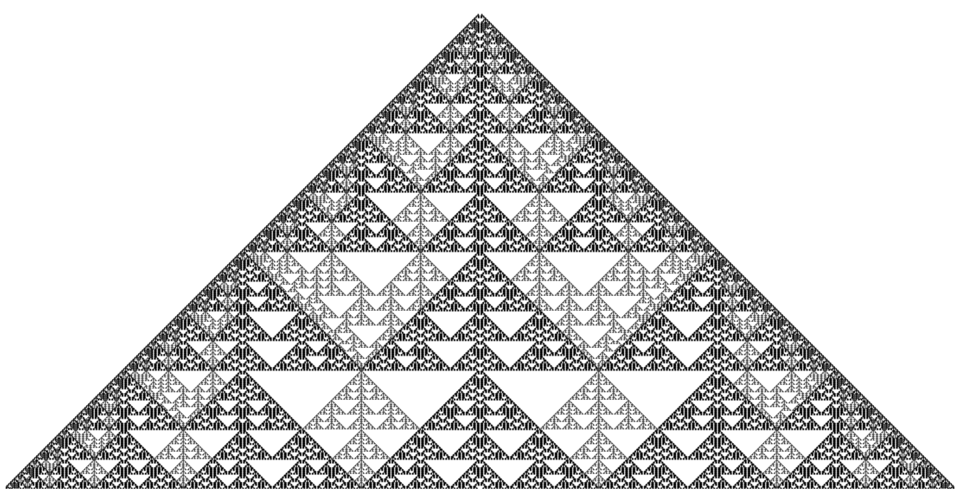

Fig. 14. *Extended 1 Or 3* from $\xi_0 = 30003$ at time 512; 3's are black, 2's are dark gray and (the rare and almost invisible) 1's are light gray.

Step 2. Starting from the same initial state as in Step 1, $\lambda_j(x) = 0$ at all even times $j \le 2^n - 4$ and $x = \pm(2^n - j - 3)$.

This follows from cancellative duality and the fact that, for even j, $\lambda_j^{\{0\}}$ occupies only even positions — in fact, by additivity, its 1's are located at twice the locations of 1's in $\lambda_{j/2}^{\{0\}}$.

These two steps already show that the Willson limit, if it exists, must be fractal with the same dimension as that of *1 Or 3*, $\log(1+\sqrt{5})/\log 2$. Indeed, the *union* of all the sets generated from all the nuclei within the relevant triangles, together with the set generated by ξ_t mod 2, is such a fractal and is an upper bond for our Willson limit. Also, a similar argument as in the proof of Theorem 5 demonstrates non-replication.

To demonstrate the existence of Willson limit and thus complete the argument, we need to specify a recursive description of the state within the relevant triangles. After a proper space and time rescaling, this reduces to the scheme for α_n described in the next step.

region	top/bottom	left	right	middle
α^{00}	α^{11}	ν	α^{00}	γ^{1000}
α^{11}	α^{11}	ν	α^{10}	γ^{1110}
α^{10}	α^{01}	ν	α^{00}	γ^{1100}
α^{01}	α^{01}	ν	α^{10}	γ^{1010}
γ^{1000}	σ^{001}	σ^{111}	γ^{1111}	ν
γ^{1111}	σ^{011}	σ^{100}	γ^{0111}	0
γ^{0111}	σ^{011}	σ^{010}	γ^{0111}	0
γ^{1110}	σ^{011}	σ^{100}	γ^{0101}	0
γ^{0101}	σ^{011}	σ^{010}	γ^{0001}	0
γ^{0001}	σ^{001}	σ^{001}	γ^{1101}	ν
γ^{1101}	σ^{011}	σ^{100}	γ^{0001}	0
γ^{1100}	σ^{011}	σ^{100}	γ^{0011}	0
γ^{0011}	σ^{001}	σ^{001}	γ^{1011}	ν
γ^{1011}	σ^{001}	σ^{111}	γ^{1011}	ν
γ^{1010}	σ^{001}	σ^{111}	γ^{1001}	ν
γ^{1001}	σ^{001}	σ^{111}	γ^{1101}	ν
ν	ν	0	0	σ^{010}
σ^{001}	0	0	σ^{011}	0
σ^{011}	σ^{010}	σ^{011}	σ^{101}	ν
σ^{111}	σ^{010}	σ^{101}	σ^{101}	ν
σ^{010}	σ^{010}	σ^{011}	σ^{110}	ν
σ^{101}	0	σ^{110}	σ^{011}	0

Table 9. Recursive specification for the defined configurations; σ^{100} and σ^{110} are given by symmetry and the four missing γ's never appear.

Step 3. Assume that $n \ge 2$; we will suppress the subscript n, which we will call *level*, from α and all other configurations and regions.

Let $\alpha = \alpha^{00}$ be the final space time configuration within $T' = \{(x,t) : |x| \le 2^n - t - 1\}$, started from $A = A_{00}$ with 1's at $2^n - 3 \cdot 2^k$, $k = 0, 1, \dots, n-2$. For $c, d \in \{0,1\}$, let A_{cd} be the configuration obtained by changing the states of A_{00} at $2^n - 3 \cdot 2^{n-3} - 2^{n-3}$ to c and at $2^n - 3 \cdot 2^{n-3} + 2^{n-3}$ to d. Then let α^{cd} be the configuration within T' started from A_{cd}. Moreover, let ν be the configuration generated within T' by a single 1 at the origin.

Let $a, b, c, d \in \{0,1\}$, and build an initial configuration as follows. First take A_{cd} *at level* $n-1$ and translate it by 2^{n-1} (that is, add 2^{n-1} to the positions of all 1's). Further, make a the state at -2^n and b the state at the origin, with 0's elsewhere. Then let γ^{abcd} be the resulting space-time configuration within $T = \{(x,t) : 0 \le t \le 2^n - 1, |x| \le t\}$. Finally, define σ^{abc} within T, for $a, b, c \in \{0,1\}$, as the eight configurations generated by a at -2^n, b at 0, c at 2^n, and 0's elsewhere. (We write 0 in place of σ^{000}.)

We divide T and T' into four triangles as in Fig. 11 (specifically, as in the α_n and σ_n division, respectively, in that figure). Then the recursions are specified in Table 9, with the four subtriangles referred to as left, right, middle, and top (for T) or bottom (for T'); all are easy to check using additivity. □

8 Open problems

(1) Does *Exactly 1* have a replicator with a nonzero ether? Does it have a replicator which emulates an additive rule different from *Rule 90*? Does it have a quasireplicator which is not a replicator?

(2) Does *Perturbed Exactly 1* have a periodic attractor? The methods developed in [**GG4**] do not apply.

(3) For *1 Or 2*, choose a random seed of a large length $n+2$. Does t_a/n converge in distribution along every subsequence $n_k = \alpha 2^k$, $\alpha \in (0,1]$? What is the asymptotic behavior of $\max t_a$? The same questions can be asked about t_0.

(4) Is there a nontrivial upper bound for replicator entropy for *Perturbed Exactly 1*? An easier task may be to prove that replicator probability does not approach 1 for long seeds. For *Exactly 1*, the latter claim follows from the existence of robust periodic solutions [**GG4**], but again the method does not extend to *Perturbed Exactly 1*.

(5) For *Embossed Triangles*, are all seeds replicators, possibly mixed with two ethers?

(6) What can be said about replication in *Quota* with arbitrary range and θ [**CD**]?

(7) Does *Perturbed Exactly 1* have a quasireplicator with the Willson limit different from the two discussed in Section 6? Can one develop a method robust enough to deal with the dynamics of Fig. 12?

(8) What is the set of possible ethers for *Extended 1 or 3*? Starting this CA from a random seed of length at most $n+2$, does replication probability approach 1 as $n \to \infty$?

References

[**BP**] E. Bilotta, P. Pantano, *Emergent patterning phenomena in 2D cellular automata*, Artificial Life 11 (2005), 339–362.

[**BM**] A. Berlinkov, R. D. Mauldin, *Packing measure and dimension of random fractals*, Journal of Theoretical Probability 15 (2002), 695–713.

[**BDR**] C. D. Brummitt, H. Delventhal, M. Retzlaff, *Packard snowflakes on the von Neumann neighborhood*, Journal of Cellular Automata 3 (2008), 57–80.

[**CD**] K. Culik, S. Dube, *Fractal and Recurrent Behavior of Cellular Automata*, Complex Systems 3 (1989), 253–267.

[**EN**] K. Eloranta, E. Nummelin, *The kink of cellular automaton Rule 18 performs a random walk*, Journal of Statistical Physics 69 (1992), 1131-1136.

[**Epp**] D. Eppstein, *Cellular Automata: Replicators*.
`http://www.ics.uci.edu/ eppstein/ca/replicators/`

[**Eva1**] K. M. Evans, "Larger than Life; it's so nonlinear." Ph. D. Thesis, University of Wisconsin, Madison, 1996.

[**Eva2**] K. M. Evans, *Replicators and Larger than Life examples*, in "New Constructions in Cellular Automata," eds. D. Griffeath and C. Moore, Oxford University Press, 2003, 119–159.

[**FLM**] P. Favatia, G. Lotti, L. Margara, *Additive one-dimensional cellular automata are chaotic according to Devaney's definition of chaos*, Theoretical Computer Science 174 (1997), 157–170.

[**GG1**] J. Gravner, D. Griffeath, *Cellular automaton growth on* $\mathbb{Z}^2$*: theorems, examples and problems*, Advances in Applied Mathematics 21 (1998), 241–304.

[**GG2**] J. Gravner, D. Griffeath, *Modeling snow crystal growth I: rigorous results for Packard's digital snowflakes*, Experimental Mathematics 15 (2006), 421-444.

[**GG3**] J. Gravner, D. Griffeath, *Asymptotic densities for Packard Box rules*, Nonlinearity 22 (2009), 1817–1846.

[**GG4**] J. Gravner, D. Griffeath, *The one-dimensional Exactly 1 cellular automaton: replication, periodicity, and chaos from finite seeds*, Journal of Statistical Physics 142 (2011), 168–200.

[**GGM**] J. Gravner, G. Gliner, M. Pelfrey, *One-dimensional replicators: examples*.
`http://math.udavis.edu/~gravner/replicators.html`

[**Gri**] D. Griffeath, "Additive and Cancellative Interacting Particle Systems." Springer, 1979.

[**HC**] J. E. Hanson, J. P. Crutchfield, *The Attractor-Basin Portrait of a Cellular Automaton*, Journal of Statistical Physics 66 (1992) 1415-1462.

[**HPS**] F. v. Haeseler, H.-O. Peitgen, G. Skordev, *Global analysis of self-similarity features of cellular automata: selected examples*, Physica D 86 (1995), 64–80.

[**Jen1**] E. Jen, *Enumeration of preimages of cellular automata*, Complex Systems 3 (1989), 421–456.

[**Jen2**] E. Jen, *Aperiodicity in one-dimensional cellular automata*, Physica D 45 (1990), 3–18.

[**Jen3**] E. Jen, *Exact solvability and quasiperiodicity of one-dimensional cellular automata*, Nonlinearity 4 (1991), 251–276.

[**Jen4**] E. Jen, *Transience and dislocations in one-dimensional cellular automata*. In "Cellular automata and cooperative systems (Les Houches, 1992)," NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 396, Kluwer Acad. Publ., 1993, 299–310.

[**MW**] R. D. Mauldin, S. C. Williams, *Hausdorff Dimension in Graph Directed Constructions*, Transactions of the American Mathematical Society, 309(1988), 811–829.

[**Moo**] C. Moore, *Predicting non-linear cellular automata quickly by decomposing them into linear ones*, Physica D 111 (1998), 27–41.

[**Oll**] N. Ollinger, *The intrinsic universality problem of one-dimensional cellular automata*. In "Symposium on Theoretical Aspects of Computer Science (STACS 2003)," eds. H. Alt and M. Habib, Springer, Berlin, 2003, 632-641).

[**Wil1**] S. J. Willson, *Cellular automata can generate fractals*, Discrete Appl. Math. 8 (1984), 91–99.

[**Wil2**] S. J. Willson, *Computing fractal dimensions for additive cellular automata*, Physica D 24 (1987), 190–206.

[**Woj**] M. Wójtowicz, *Mirek's Cellebration: a 1D and 2D Cellular Automata explorer*. `http://www.mirwoj.opus.chelm.pl/ca/`

[**Wol1**] S. Wolfram, *Statistical Mechanics of Cellular Automata*, Rev. Mod. Phys. 55 (1983) 601–644.

[**Wol2**] S. Wolfram, *Computational Theory of Cellular Automata*, Communications in Mathematical Physics 96 (1984), 15–57.

[**Wol3**] S. Wolfram, "A New Kind of Science." Wolfram Media, 2002.