Math 118 Fall 2013

Homework 5 Solutions

2.3.2 (a) increases or stays the same. (b) decreases or stays the same.

- Consider the two rectangles $R_1 := \{0 \le x \le l,\ 0 \le t \le T_1\}$ and $R_2 := \{0 \le x \le l,\ 0 \le t \le T_2\}$ with $T_1 < T_2$. We have $R_1 \subset R_2$. Thus, $M(T_1) \le M(T_2)$ and $m(T_1) \ge m(T_2)$.

2.3.5

- Given $u = -2xt - x^2$, we have $u_t = -2x = xu_{xx}$, so it is a solution.
-
  - On $x = -2$: $u = 4t - 4$. The max occurs at $t = 1$ with $u(-2, 1) = 0$.
  - On $x = 2$: $u = -4t - 4$. The max occurs at $t = 0$ with $u(2, 0) = -4$.
  - On $t = 0$: $u = -x^2$. The max occurs at $x = 0$ with $u(0, 0) = 0$.
  - On $t = 1$: $u = -2x - x^2 = 1 - (x + 1)^2$. The max occurs at $x = -1$ with $u(-1, 1) = 1$.
- The maximum is assumed on the top instead of on the bottom or the lateral sides, which contradicts the Maximum Principle.

2.3.6

- Consider $\omega = u - v$, then $\omega \le 0$ for $t = 0, x = 0$ and $x = l$.
- By the Maximum Principle, the maximum of $\omega(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$). Denote the maximum by $\omega_{\max}$. We then have $\omega_{\max} \le 0$.
- It follows that $\omega(x, t) \le \omega_{\max} \le 0$, i.e., $u \le v$, for $0 \le t < \infty, 0 \le x \le l$.

2.3.8

- Multiplying the diffusion equation by u, we obtain
$$uu_t = ku_{xx}u \iff (\frac{1}{2}u^2)_t = (kuu_x)_x - ku_x^2.$$
- Upon integrating over the interval $0 < x < l$, we get
$$\frac{1}{2}\left(\frac{d}{dt}\int_0^l u^2\,dx\right) = \int_0^l (\frac{1}{2}u^2)_t\,dx = (kuu_x)_x|_{x=0}^{x=l} - k\int_0^l u_x^2\,dx.$$
- Plugging in the Robin boundary conditions, we have
$$\frac{1}{2}\left(\frac{d}{dt}\int_0^l u^2\,dx\right) = -a_l u^2(l, t) - a_0 u^2(0, t) - k\int_0^l u_x^2\,dx.$$
- Since $a_0 > 0$ and $a_l > 0$, the endpoints contribute to the decrease of $\int_0^l u^2(x, t)\,dx$.
- It's easy to verify that the right hand side is indeed negative.

2.5.1

- Consider the "hammer blow" example, problem 4 and 5 from section 2.4, as a counter example.