Final: Solutions
Math 118A, Fall 2013

1. [20 pts] For each of the following PDEs for $u(x, y)$, give their order and say if they are nonlinear or linear. If they are linear, say if they are homogeneous or nonhomogeneous and if they have constant or variable coefficients.

(a) $u_x = (\sin x)u_y$

(b) $uu_x + u_y = u_{xx} + \sin x$

(c) $u_{xxyy} = \sin x$

Solution.

- (a) 1st order, linear, homogeneous, variable coefficient.
- (b) 2nd order, nonlinear.
- (c) 4th order, linear, nonhomogeneous, constant coefficient.

2. [30 pts] Solve the following initial value problem for $u(x,t)$:

$$u_t + 3u_x = \sin t, \qquad u(x,0) = \sin x.$$

Solution.

- The PDE has particular solutions $u = u_p(t)$ depending only on t, where

$$\frac{du_p}{dt} = \sin t.$$

  For example, we can take $u_p(t) = -\cos t$.

- Writing $u(x,t) = u_p(t) + v(x,t)$, and using the linearity of the PDE, we find that v satisfies

$$v_t + 3v_x = 0, \qquad v(x,0) = \sin x + 1.$$

- The solution of this IVP for an advection equation with speed 3 is

$$v(x,t) = \sin(x - 3t) + 1.$$

- The solution of the original IVP is

$$u(x,t) = 1 - \cos t + \sin(x - 3t).$$

3. [30 pts] (a) Solve the following initial-boundary value problem for the heat equation for $u(x,t)$:

$$\begin{aligned} &u_t = u_{xx} && 0 < x < 1, \quad t > 0, \\ &u_x(0,t) = 0, \quad && u_x(1,t) = 0, \qquad t > 0, \\ &u(x,0) = f(x) && 0 \le x \le 1. \end{aligned}$$

(b) What type of boundary conditions are these? How does your solution behave as $t \to +\infty$? Give a physical explanation of this behavior.

Solution.

- (a) Separation of variables for the heat equation with Neumann BCs gives the separated solutions (derivation is omitted)

$$u(x,t) = \cos(n\pi x)e^{-n^2\pi^2 t}, \qquad n = 0, 1, 2, \dots.$$

- Taking a linear superposition of these solutions, we find that the general solution of the PDE and the BCs is

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)e^{-n^2\pi^2 t},$$

  where the a_n are arbitrary constants.

- The initial condition is satisfied if

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = f(x),$$

  meaning that the a_n are the Fourier cosine coefficients of $f(x)$, which are given by

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx, \qquad n = 0, 1, 2, \dots.$$

- (b) The BCs are Neumann BCs. We have

$$u(x,t) \to \frac{1}{2}a_0 = \int_0^1 f(x)\,dx \qquad \text{as } t \to \infty.$$

  The problem describes heat flow in a fully insulated rod. The time-asymptotic state is a uniform temperature distribution with the same thermal energy as the non-uniform initial data.

4. [30 pts] Use separation of variables to solve the following Dirichlet problem for Lapace's equation in polar coordinates for $u(r,\theta)$ in the unit disc $r < 1$:

$$\begin{aligned} &\frac{1}{r}\left(r u_r\right)_r + \frac{1}{r^2} u_{\theta\theta} = 0, \\ &u(1,\theta) = 1 \qquad \text{if } 0 < \theta < \pi, \\ &u(1,\theta) = -1 \qquad \text{if } \pi < \theta < 2\pi. \end{aligned}$$

Solution.

- The separated solutions of Lapace's equation in polar coordinates that are continuous at $r = 0$ and 2π-periodic in θ are (derivation is omitted)

$$u(r,\theta) = 1, \qquad u(r,\theta) = \begin{cases} r^n \cos(n\theta) \\ r^n \sin(n\theta) \end{cases} \qquad n = 1, 2, 3, \ldots.$$

- Taking a linear superposition of these solutions, we find that the general solution of the PDE is given by

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{ a_n r^n \cos n\theta + b_n r^n \sin n\theta \right\}$$

  where the a_n, b_n are arbitrary constants.

- Imposing the BC at $r = 1$, we get that

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos n\theta + b_n \sin n\theta \right\} = f(\theta),$$

  meaning that a_n, b_n are the full Fourier coefficients of f, which are given by

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta.$$

- The given boundary data has an odd 2π-periodic extension, so $a_n = 0$ for all n, and

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \sin n\theta \, d\theta \\ &= \frac{2}{n\pi}\left[-\cos n\theta\right]_0^{\pi} \\ &= \frac{2}{n\pi}\left[1 - \cos n\pi\right] \\ &= \frac{2}{n\pi}\left[1 - (-1)^n\right] \\ &= \begin{cases} 4/n\pi & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

- The solution is

$$u(x,t) = \sum_{n \text{ odd}} \frac{4}{n\pi} r^n \sin n\theta.$$

5. [25 pts] (a) For all (smooth) functions $X(x)$, $Y(x)$, prove that

$$\int_a^b (XY'' - YX'')\,dx = [XY' - YX']_a^b\,.$$

(b) Suppose that $X_1(x)$, $X_2(x)$ are solutions of the eigenvalue problem

$$\begin{aligned} -X_1'' &= \lambda_1 X_1, \qquad X_1(a) = 3X_1(b), \quad 3X_1'(a) = X_1'(b), \\ -X_2'' &= \lambda_2 X_2, \qquad X_2(a) = 3X_2(b), \quad 3X_2'(a) = X_2'(b), \end{aligned}$$

where $\lambda_1 \neq \lambda_2$ are distinct, real eigenvalues. Show that X_1 and X_2 are orthogonal, meaning that $\int_a^b X_1 X_2\,dx = 0$.

Solution.

- (a) By the product rule,

$$\begin{aligned} [XY' - YX']' &= XY'' + X'Y' - (YX'' + Y'X') \\ &= XY'' - YX'', \end{aligned}$$

  so the result follows from the fundamental theorem of calculus.

- (b) Using the ODEs for X_1, X_2, we have

$$\int_a^b (X_1 X_2'' - X_2 X_1'')\,dx = (\lambda_1 - \lambda_2) \int_a^b X_1 X_2\,dx.$$

  It then follows from the identity in (a) that

$$(\lambda_1 - \lambda_2) \int_a^b X_1 X_2\,dx = [X_1 X_2' - X_2 X_1']_a^b\,.$$

- The boundary conditions satisfied by X_1, X_2 imply that

$$\begin{aligned} X_1(b)X_2'(b) - X_2(b)X_1'(b) &= \frac{1}{3}X_1(a) \cdot 3X_2'(a) - \frac{1}{3}X_2(a) \cdot 3X_2'(a) \\ &= X_1(a)X_2'(a) - X_2(a)X_1'(a) \end{aligned}$$

- It follows that the boundary terms cancel, so

$$(\lambda_1 - \lambda_2) \int_a^b X_1 X_2\,dx = 0,$$

  which implies that X_1 and X_2 are orthogonal if $\lambda_1 \neq \lambda_2$.

6. [25 pts] Let Ω be a bounded open set in $\mathbb{R}^2$ with boundary $\partial\Omega$.

(a) Suppose that $u(x, y)$ is a solution of the PDE

$$u_{xx} + u_{yy} - u = 0.$$

Show that u cannot attain a maximum value at any point of Ω where $u > 0$, or a minimum value at any point of Ω where $u < 0$.

(b) Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be given functions. Show that a solution of the following Dirichlet boundary value problem is unique:

$$\begin{aligned} &u_{xx} + u_{yy} - u = f \qquad \text{in } \Omega, \\ &u = g \qquad \text{on } \partial\Omega, \end{aligned}$$

Solution.

- (a) Suppose that u attain a (local) maximum at some point in Ω. Since Ω is open, the maximum is attained at an interior point, and the second derivative test implies that $u_{xx} \leq 0$ and $u_{yy} \leq 0$ at this point. It follows from the PDE that $u = u_{xx} + u_{yy} \leq 0$, so u cannot attain a maximum at any point where $u > 0$. Similarly, at a minimum we have $u_{xx} \geq 0$ and $u_{yy} \geq 0$, so $u = u_{xx} + u_{yy} \geq 0$, and u cannot attain a minimum at any point in Ω where $u < 0$.

- (b) Suppose that u_1, u_2 are solutions of the BVP. Let $v = u_1 - u_2$. Then, by linearity,

$$v_{xx} + v_{yy} - v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

  Since $\bar{\Omega} = \Omega \cup \partial\Omega$ is closed and bounded, and a solution v is assumed to be continuous on $\bar{\Omega}$, v attains its maximum value $M = \max_{\bar{\Omega}} v$ at some point in $\bar{\Omega}$. If $M > 0$, then (since $v = 0$ on $\partial\Omega$) the maximum would have to be attained at an interior point in Ω where $v = M > 0$, contradicting (a). Similarly, if $m = \min_{\bar{\Omega}} v < 0$, then the mimimum would have to be attained at an interior point in Ω where $v = m < 0$, also contradicting (a). It follows that the maximum and minimum are attained on the boundary $\partial\Omega$, so $m = M = 0$, which implies that $v = 0$, and $u_1 = u_2$.

Remark. This argument doesn't work for the PDE $u_{xx} + u_{yy} + u = 0$, with the opposite sign on u. In that case, the Dirichlet problem might have non-zero solutions. This corresponds to the fact that the eigenvalues of the Dirichlet problem $-\Delta u = \lambda u$ for the Laplacian are always positive ($\lambda > 0$).

7. [40 pts] Let c, V be positive constants, and consider the PDE

$$u_{tt} + 2V u_{xt} + \left(V^2 - c^2\right) u_{xx} = 0.$$

(a) Show that the change of variables

$$u(x,t) = w(\xi, \tau), \qquad \xi = x - Vt, \quad \tau = t$$

transforms the PDE into the wave equation $w_{\tau\tau} - c^2 w_{\xi\xi} = 0$.

(b) Solve the initial value problem

$$\begin{aligned}
&u_{tt} + 2V u_{xt} + \left(V^2 - c^2\right) u_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0,\\
&u(x,0) = \phi(x), \qquad -\infty < x < \infty,\\
&u_t(x,0) = \psi(x), \qquad -\infty < x < \infty.
\end{aligned}$$

(c) Describe the domains of dependence and influence for this PDE and sketch them in the (x,t)-plane. Consider the cases: (i) $0 < V < c$; (ii) $0 < c < V$.

Solution.

- (a) By the chain rule for partial derivatives,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - V\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}$$

  It follows that $u_{xx} = w_{\xi\xi}$ and

$$\begin{aligned}
u_t &= w_\tau - V w_\xi, \qquad ,\\
u_{xt} &= w_{\xi\tau} - V w_{\xi\xi},\\
u_{tt} &= w_{\tau\tau} - V w_{\tau\xi} - V\left(w_{\xi\tau} - V w_{\xi\xi}\right)\\
&= w_{\tau\tau} - 2V w_{\xi\tau} + V^2 w_{\xi\xi}.
\end{aligned}$$

- Using these expressions in the left-hand side of the PDE and simplifying the result, we get

$$\begin{aligned}
&u_{tt} + 2V u_{xt} + \left(V^2 - c^2\right) u_{xx}\\
&\quad = w_{\tau\tau} - 2V w_{\xi\tau} + V^2 w_{\xi\xi} + 2V\left(w_{\xi\tau} - V w_{\xi\xi}\right) + \left(V^2 - c^2\right) w_{\xi\xi}\\
&\quad = w_{\tau\tau} - c^2 w_{\xi\xi}.
\end{aligned}$$

  It follows that $w_{\tau\tau} - c^2 w_{\xi\xi} = 0$.

- (b) Transforming the IVP from u to w, we find that $w(\xi, \tau)$ satisfies

$$\begin{aligned}
&w_{\tau\tau} - c^2 w_{\xi\xi} = 0, \qquad -\infty < \xi < \infty, \quad \tau > 0,\\
&w(\xi, 0) = \phi(\xi), \qquad -\infty < \xi < \infty,\\
&w_\tau(\xi, 0) = \tilde{\psi}(\xi), \qquad -\infty < \xi < \infty,
\end{aligned}$$

  where

$$\tilde{\psi}(\xi) = \psi(\xi) + V\phi'(\xi).$$

  To derive the initial conditions, note that $x = \xi$ at $t = \tau = 0$, so $w(\xi, 0) = u(\xi, 0) = \phi(\xi)$, and from the change of variables

$$w_\tau(\xi, 0) = u_t(\xi, 0) + V w_\xi(\xi, 0) = \psi(\xi) + V\phi'(\xi),$$

  where the prime denotes a ξ-derivative.

- From d'Alembert's solution, the solution of this IVP for $w(\xi, \tau)$ is

$$w(\xi, \tau) = \frac{1}{2}\left[\phi(\xi - c\tau) + \phi(\xi + c\tau)\right] + \frac{1}{2c}\int_{\xi - c\tau}^{\xi + c\tau} \tilde{\psi}(s)\, ds.$$

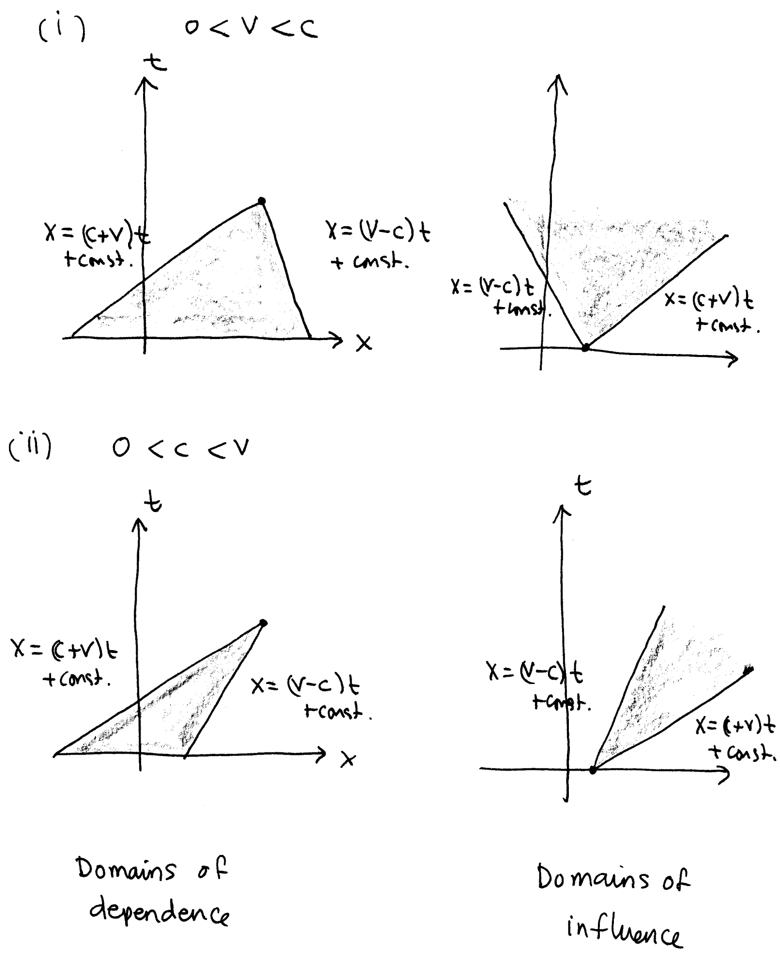
- The corresponding solution for $u(x, t) = w(x - Vt, t)$ is

$$u(x, t) = \frac{1}{2}\left[\phi(x - (c + V)t) + \phi(x + (c - V)t)\right] + \frac{1}{2c}\int_{x-(c+V)t}^{x+(c-V)t} \tilde{\psi}(s)\, ds.$$

- Using the expression for $\tilde{\psi}$ in this equation and integrating the term proportional to ϕ' in the result, we can write this solution as

$$\begin{aligned}
u(x, t) = \frac{1}{2}&\left[\left(1 - \frac{V}{c}\right)\phi(x - (c + V)t) + \left(1 + \frac{V}{c}\right)\phi(x + (c - V)t)\right]\\
&+ \frac{1}{2c}\int_{x-(c+V)t}^{x+(c-V)t} \psi(s)\, ds.
\end{aligned}$$

- (c) Domains of dependence and influence are shown on the next page.



Remark. This wave equation describes, for example, sound waves in a fluid with sound speed c that is moving with speed V. The sound waves propagate against and in the same direction as the flow with speeds $V - c$ and $V + c$, respectively. If $0 < V < c$ (subsonic flow with Mach number $M = V/c < 1$), then the sound waves can propagate both upstream and downstream, but if $0 < c < V$ (supersonic flow with Mach number $M > 1$), then the sound waves can only propagate downstream. This explains, for example, why you can't hear a supersonic aircraft coming, because it's moving faster than the sound waves it produces.