Midterm 1: Solutions to Sample questions
Math 118B, Winter 2014

1. State Green's first and second identities. If Ω is a bounded set with smooth boundary and $\alpha > 0$, use Green's first identity to show that solutions of Poisson's equation with Robin boundary conditions,

$$\begin{aligned} -\Delta u = f \qquad & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \alpha u = g \qquad & \text{on } \partial\Omega, \end{aligned}$$

are unique.

Solution.

- Green's first and second identities omitted, but you should know them!
- By taking the difference of two solutions, we just need to show that the only solution when $f = 0$, $g = 0$ is $u = 0$. In that case, by Green's first identity,

$$\begin{aligned} \int_\Omega |\nabla u|^2 \, dV &= -\int_\Omega u\Delta u \, dV + \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, dS \\ &= -\alpha \int_{\partial\Omega} u^2 \, dS. \end{aligned}$$

- Since $\alpha > 0$, it follows that both terms must be zero, so $\nabla u = 0$ in Ω and $u = 0$ on $\partial\Omega$, which implies that $u = 0$.

2. Find the Green's function for the BVP

$$
\begin{aligned}
-u'' &= f(x) \qquad 0 < x < 1,\\
u(0) = A, &\qquad u'(1) = B.
\end{aligned}
$$

Write down the Green's function representation of the solution.

Solution.

- The Green's function $G(x;\xi)$: (i) satisfies the homogeneous ODE

$$
\frac{d^2G}{dx^2} = 0 \qquad \text{for } x \neq \xi;
$$

(ii) is continuous at $x = \xi$ with a jump in its x-derivative of -1; and (iii) satisfies the homogeneous boundary conditions $G(0,\xi) = G_x(1;\xi) = 0$. It follows that

$$
G(x;\xi) = \begin{cases} x & \text{if } 0 \leq x < \xi,\\ \xi & \text{if } \xi \leq x \leq 1. \end{cases}
$$

- Using the equations for u and G in the one-dimensional form of Green's second identity,

$$
\begin{aligned}
\int_0^1 G(x;\xi)\frac{d^2u}{dx^2} - u(x)\frac{d^2G}{dx^2}(x;\xi)\,dx&\\
= \left[G(x;\xi)\frac{du}{dx}(x) - u(x)\frac{dG}{dx}(x;\xi)\right]_{x=0}^1&,
\end{aligned}
$$

and evaluating the resulting δ-function integral and boundary terms, we get

$$
-\int_0^1 G(x;\xi)f(x)\,dx + u(\xi) = BG(1;\xi) + AG_x(0;\xi),
$$

so

$$
\begin{aligned}
u(\xi) &= \int_0^1 G(x;\xi)f(x)\,dx + B\xi + A\\
&= \int_0^\xi xf(x)\,dx + \xi\int_\xi^1 f(x)\,dx + B\xi + A.
\end{aligned}
$$

(For example, if $f = 0$, we get the linear solution $u(x) = Bx + A$.)

3. Suppose that $u(\vec{x})$ is the steady temperature distribution of a body, whose heat energy density is proportional to temperature, and $\vec{q}(\vec{x})$ the heat flux vector. If there are no internal heat sources and $\vec{q} = -A\nabla u$ where A is a symmetric matrix, write down the integral form of conservation of energy and derive a PDE for u.

Remark. This constitutive relative for the flux describes anisotropic materials, in which case the heat flux needn't be in the same direction as the temperature gradient.

Solution.

- Conservation of energy implies that, in a steady state, the net energy flux out of a volume Ω is zero, so

$$\int_{\partial\Omega} \vec{q} \cdot \vec{n}\, dS = 0.$$

- The divergence theorem implies that

$$\int_{\Omega} \operatorname{div} \vec{q}\, dV = 0$$

  so since Ω is arbitrary, and assuming that $\operatorname{div} \vec{q}$ is continuous, we must have

$$\operatorname{div} \vec{q} = 0.$$

- It follows that u satisfies the PDE

$$\operatorname{div} (A\nabla u) = 0.$$

  (If A is the identity matrix, this is just Laplace's equation.)

4. Let $G(\vec{x})$ be the free-space Green's function for the Helmholtz equation

$$-\Delta G + G = \delta(\vec{x}), \qquad G(\vec{x}) \to 0 \quad \text{as } |\vec{x}| \to \infty$$

in three space dimensions. Write down the conditions that determine G, and solve for G. Write down the Green's function representation of the solution of

$$-\Delta u + u = f(\vec{x}), \qquad u(\vec{x}) \to 0 \quad \text{as } |\vec{x}| \to \infty$$

where $f(\vec{x})$ is a smooth function that is zero when $|\vec{x}|$ is sufficiently large.
Hint. The three-dimensional Laplacian of functions $u(r)$ of $r = |\vec{x}|$ is given by

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$

Write $G = H/r$ and solve for H.

Solution.

- We require that: (i) $-\Delta G + G = 0$ if $\vec{x} \neq 0$; (ii) G is integrable with

$$\begin{aligned}
\lim_{\epsilon \to 0^+} \int_{\partial B_\epsilon(0)} \frac{\partial G}{\partial n}\, dS &= \lim_{\epsilon \to 0^+} \int_{B_\epsilon(0)} \Delta G \, d\vec{x} \\
&= \lim_{\epsilon \to 0^+} \int_{B_\epsilon(0)} \{\Delta G - G\}\, d\vec{x} \\
&= -\lim_{\epsilon \to 0^+} \int_{B_\epsilon(0)} \delta(\vec{x})\, d\vec{x} \\
&= -1,
\end{aligned}$$

  where $B_\epsilon(0)$ is the ball of radius ϵ centered at 0, and $\partial B_\epsilon(0)$ is the sphere. (Here, we use a formal δ-function calculation, and $\partial/\partial n = \partial/\partial r$ is the outward normal derivative to $B_\epsilon(0)$.)

- Assuming that $G = G(r)$ is a spherically symmetric function, the ODE for G in $r > 0$ is

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dG}{dr}\right) + G = 0.$$

- Writing $G(r) = H(r)/r$ and simplifying the result, we find that $H(r)$ satisfies

$$\frac{d^2H}{dr^2} - H(r) = 0.$$

  Since $G(r) \to 0$ as $r \to \infty$, we have $H(r) = Ce^{-r}$ for some constant C, and $G(r) = Ce^{-r}/r$.

- We then find that

$$\begin{aligned}\lim_{\epsilon\to 0^+} \int_{\partial B_\epsilon(0)} \frac{\partial G}{\partial n}\, dS &= \lim_{\epsilon\to 0^+} 4\pi\epsilon^2 \cdot \left.\frac{dG}{dr}\right|_{r=\epsilon} \\ &= \lim_{\epsilon\to 0^+} 4\pi\epsilon^2 \cdot C\left(-\frac{e^{-\epsilon}}{\epsilon^2} - \frac{e^{-\epsilon}}{\epsilon}\right) \\ &= -4\pi C\end{aligned}$$

  so $4\pi C = 1$ and

$$G(\vec{x}) = \frac{e^{-|\vec{x}|}}{4\pi|\vec{x}|}$$

- The Green's function representation of the solution for u is

$$u(\vec{x}) = \frac{1}{4\pi}\int \frac{e^{-|\vec{x}-\vec{\xi}|}}{|\vec{x}-\vec{\xi}|} f(\vec{\xi})\, d\vec{\xi}.$$