Problem Set 1: Solutions
Math 201A: Fall 2016

Problem 1. Let (X, d) be a metric space.
(a) Prove the reverse triangle inequality: for every $x, y, z \in X$

$$d(x, y) \geq |d(x, z) - d(z, y)|.$$

(b) Prove that if $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$.

Solution

- (a) The triangle inequality

  $$d(x, y) + d(y, z) \geq d(x, z)$$

  implies that
  $$d(x, y) \geq d(x, z) - d(y, z).$$
  Exchanging x and y, and using the symmetry of d, we also have

  $$d(x, y) \geq d(y, z) - d(x, z).$$

  Hence
  $$d(x, y) \geq |d(x, z) - d(y, z)|.$$

- (b) Using the reverse triangle inequality, we get that

  $$\begin{aligned} |d(x_n, y_n) - d(x, y)| &\leq |d(x_n, y_n) - d(x, y_n)| + |d(x, y_n) - d(x, y)| \\ &\leq d(x_n, x) + d(y_n, y) \\ &\to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

Problem 2. Let E be a finite set and let $P = \mathcal{P}(E)$ be the power set of E (the set of all subsets of E). Define $d : P \times P \to \mathbb{R}$ by

$$d(A, B) = \operatorname{card}(A \Delta B)$$

where $\operatorname{card}(A)$ is the number of elements of A and

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

is the symmetric difference of $A, B \subset E$. Show that (P, d) is a metric space.

Solution

- We have $d(A, B) \geq 0$. If $d(A, B) = 0$, then $A \setminus B = A \cap B^c = \emptyset$, so $B \supset A$. Similarly, $A \supset B$, so $A = B$.

- The symmetry of d is immediate.

- Let $A, B, C \subset X$. Then

  $$\begin{aligned} A \Delta B &= (A \cap B^c) \cup (A^c \cap B) \\ &= (A \cap B^c \cap C) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C) \cup (A^c \cap B \cap C^c) \\ &= F \cup G, \end{aligned}$$

  where (draw a Venn diagram!)

  $$\begin{aligned} F &= (A^c \cap B \cap C) \cup (A \cap B^c \cap C^c), \\ G &= (A \cap B^c \cap C) \cup (A^c \cap B \cap C^c). \end{aligned}$$

- If $x \in F$, then either $x \in A^c \cap B$ and $x \in C$, which implies that $x \notin G$, or $x \in A \cap B^c$ and $x \in C^c$, which also implies that $x \notin G$. It follows that $F \cap G = \emptyset$ and

  $$\operatorname{card}(A \Delta B) = \operatorname{card}(F) + \operatorname{card}(G).$$

- We have

  $$F \subset (A^c \cap C) \cup (A \cap C^c) = A \Delta C,$$

  so $\operatorname{card}(F) \leq \operatorname{card}(A \Delta C)$. Similarly, $\operatorname{card}(G) \leq \operatorname{card}(B \Delta C)$, which shows that

  $$\operatorname{card}(A \Delta B) \leq \operatorname{card}(A \Delta C) + \operatorname{card}(B \Delta C).$$

  Thus, d satisfies the triangle inequality.

Remark. In coding theory, d is called the Hamming metric, which measures the number of mismatches between two finite strings of 0s and 1s.

Problem 3. If (X, d) is a metric space, define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

(a) Show that (X, ρ) is a metric space.
(b) Show that (X, d) and (X, ρ) have the same convergent sequences and the same metric topologies. Do they necessarily have the same Cauchy sequences?

Solution

- (a) Let $s, t \geq 0$. Then

  $$\frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t}.$$

  Moreover,

  $$\frac{s}{1+s} - \frac{t}{1+t} = \frac{s-t}{(1+s)(1+t)},$$

  so $0 \leq t \leq s$ implies that

  $$\frac{t}{1+t} \leq \frac{s}{1+s}$$

- The positivity and symmetry of ρ are immediate.

- Let $x, y, z \in X$. Using the triangle inequality for d and the previous inequalities, we get that

  $$\begin{aligned}\rho(x, y) &= \frac{d(x, y)}{1 + d(x, y)} \\ &\leq \frac{d(x, z) + d(y, z)}{1 + d(x, z) + d(y, z)} \\ &\leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(y, z)}{1 + d(y, z)} \\ &\leq \rho(x, z) + \rho(y, z),\end{aligned}$$

  so ρ satisfies the triangle inequality, and (X, ρ) is a metric space.

- (b) Clearly, $d(x_n, x) \to 0$ if and only if $\rho(x_n, x) \to 0$, so d and ρ have the same convergent sequences.

- Let $B_r(x)$ denote the open ball with respect to d and $C_r(x)$ the open ball with respect to ρ. If $d(x, y) < r$, then $\rho(x, y) < r$, so $B_r(x) \subset C_r(x)$. It follows that if G is open with respect to ρ and $C_\epsilon(x) \subset G$ for each $x \in G$ and some $\epsilon > 0$, then $B_\epsilon(x) \subset G$, so G is open with respect to d.

- Similarly, if $\rho(x, y) < r$ where $r < 1/2$, then $d(x, y) < 2r$, so $C_r(x) \subset B_{2r}(x)$. If G is open with respect to d and $B_\epsilon(x) \subset G$, then we can choose $\epsilon < 1/2$ without loss of generality, and $C_{\epsilon/2}(x) \subset G$, so G is open with respect to ρ.

- The two metrics have the same Cauchy sequences. Suppose that (x_n) is Cauchy in (X, ρ) and let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\rho(x_m, x_n) < \min\left\{\frac{\epsilon}{2}, \frac{1}{2}\right\} \qquad \text{for all } m, n > N.$$

  Then $d(x_m, x_n) < \epsilon$ for all $m, n > N$, so (x_n) is Cauchy in (X, d). The converse is similar.

Problem 4. Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x, y) = \sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|} \qquad x = (x_1, x_2), \quad y = (y_1, y_2).$$

(a) Show that $(\mathbb{R}^2, d)$ is a metric space. Is this metric derived from a norm $\|\cdot\|$ on $\mathbb{R}^2$, meaning that $d(x, y) = \|x - y\|$?
(b) Sketch the unit ball $B_1(0)$ in $(\mathbb{R}^2, d)$. Is it a convex set?

Solution

- (a) The symmetry and positivity of d are immediate, so we just need to verify the triangle inequality.

- For any $a, b \geq 0$, we have

  $$\left(\sqrt{a} + \sqrt{b}\right)^2 = a + 2\sqrt{ab} + b \geq a + b,$$

  which shows that

  $$\sqrt{a} + \sqrt{b} \geq \sqrt{a + b},$$

  with equality if and only if $a = 0$ or $b = 0$.

- Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then, since $x \mapsto \sqrt{x}$ is an increasing function, the previous inequality implies that

  $$\begin{aligned} d(x, y) &= \sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|} \\ &\leq \sqrt{|x_1 - z_1| + |z_1 - y_1|} + \sqrt{|x_2 - z_2| + |z_2 - y_2|} \\ &\leq \sqrt{|x_1 - z_1|} + \sqrt{|y_1 - z_1|} + \sqrt{|x_2 - z_2|} + \sqrt{|y_2 - z_2|} \\ &\leq d(x, z) + d(z, y). \end{aligned}$$

- The metric is not derived from a norm on $\mathbb{R}^2$ since

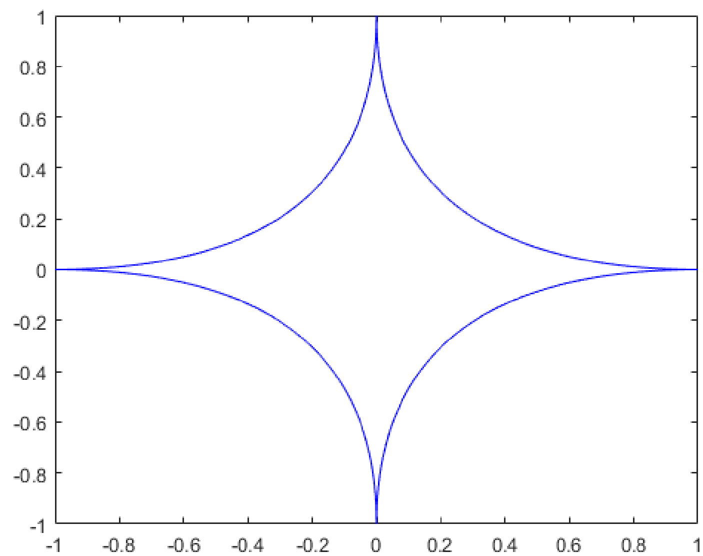
  $$d(\lambda x, \lambda y) = \sqrt{|\lambda|} d(x, y)$$

  for $\lambda \in \mathbb{R}$, so it is not homogeneous of degree one.

- (b) The unit ball is shown in the figure. It is not convex. For example, if $1/2 \leq |a| < 1$ and

  $$x = (a, 0), \qquad y = (0, a), \qquad z = \frac{1}{2}(x + y),$$

then $d(x,0) = d(y,0) = \sqrt{a} < 1$ and $d(z,0) = \sqrt{2a} \geq 1$, so $x, y \in B_1(0)$ but $z \notin B_1(0)$.



Remark. The unit ball of a (real) normed space is always convex, since $\|x\|, \|y\| < 1$ and $0 \leq \lambda \leq 1$ implies that

$$\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\| < 1.$$

Problem 5. Define the closure $\bar{A}$ of a subset $A \subset X$ of a metric space X by

$$\bar{A} = \bigcap \{F \subset X : F \supset A \text{ and } F \text{ is closed}\}.$$

Show that

$$\bar{A} = \{x \in X : \text{there exists a sequence } (x_n) \text{ with } x_n \in A \text{ and } x_n \to x\}.$$

Solution

- First, we show that $x \in \bar{A}$ if and only if every neighborhood of x contains some point in A. To do this, we prove the equivalent statement that $x \notin \bar{A}$ if and only if some neighborhood of x is disjoint from A.
- If $x \notin \bar{A}$, then since $\bar{A} \supset A$ is closed and $\bar{A}^c \subset A^c$ is open, there is a neighborhood $U_x \subset \bar{A}^c$ of x that is disjoint from A.
- Conversely, if U_x is an open neighborhood of $x \in X$ that is disjoint from A, then $F = U_x^c$ is a closed set with $F \supset A$ and $x \notin F$ so $x \notin \bar{A}$.
- Let $\tilde{A}$ denote the sequential closure of A:

  $$\tilde{A} = \{x \in X : \text{there exists a sequence } (x_n) \text{ with } x_n \in A \text{ and } x_n \to x\}.$$

- If $x \notin \bar{A}$, then x has a neighborhood that is disjoint from $\bar{A} \supset A$, so no sequence in A can converge to x and $x \notin \tilde{A}$. It follows that $\bar{A} \supset \tilde{A}$.
- If $x \in \bar{A}$, then for every $n \in \mathbb{N}$, there exists $x_n \in B_{1/n}(x) \cap A$, so (x_n) is a sequence in A that converges to x, and $x \in \tilde{A}$. It follows that $\tilde{A} \supset \bar{A}$, so $\tilde{A} = \bar{A}$.

Problem 6. Is the closure of the open ball

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

in a metric space (X, d) always equal to the closed ball

$$\bar{B}_r(x) = \{y \in X : d(x, y) \le r\}?$$

Solution

- This is not true in general.
- For example, if X is a set with at least two elements and $d : X \times X \to \mathbb{R}$ is the discrete metric,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

  then every subset of X is closed and $B_1(x) = \{x\}$, $\overline{B_1(x)} = \{x\}$, but $\bar{B}_1(x) = X$, so $\overline{B_1(x)} \neq \bar{B}_1(x)$.

Problem 7. Let X be the space of all real sequences of the form

$$x = (x_1, x_2, x_3, \dots, x_N, 0, 0, \dots) \quad \text{for some } N \in \mathbb{N}, \text{ where } x_n \in \mathbb{R},$$

whose terms are zero from some point on. Define

$$\|x\| = \max_{n \in \mathbb{N}} |x_n|.$$

(a) Show that $(X, \|\cdot\|)$ is a normed linear space (with vector addition and scalar multiplication defined componentwise).
(b) Show that X is not complete.
(c) Let c_0 denote the space of all real sequences (x_n) such that $x_n \to 0$ as $n \to \infty$. Show that $(c_0, \|\cdot\|)$ is complete and X is dense in c_0.

Solution.

- (a) It is immediate to verify that X is a linear space under componentwise addition and scalar multiplication. (Note that a finite linear combination of sequences in X also belongs to X.)
- The properties of a norm are straightforward to check. For example, if $x = (x_n)$ and $y = (y_n)$, then

$$\begin{aligned} \|x + y\| &= \max_{n \in \mathbb{N}} |x_n + y_n| \\ &\leq \max_{n \in \mathbb{N}} \{|x_n| + |y_n|\} \\ &\leq \max_{n \in \mathbb{N}} |x_n| + \max_{n \in \mathbb{N}} |y_n| \\ &\leq \|x\| + \|y\|. \end{aligned}$$

- (b) Consider the sequence $\left(x^{(k)}\right)$ in X defined for $k \in \mathbb{N}$ by

$$x^{(k)} = (1, 1/2, 1/3, \dots, 1/k, 0, 0, \dots).$$

Then, for all $j > k$, we have

$$\left\|x^{(j)} - x^{(k)}\right\| = \frac{1}{k+1},$$

so the sequence is Cauchy. However, if $x = (x_1, x_2, \dots, x_N, 0, 0, \dots)$ is any point in X, then

$$\left\|x^{(k)} - x\right\| \geq \frac{1}{N+1} \qquad \text{for all } k \geq N+1,$$

so the sequence $\left(x^{(k)}\right)$ does not have a limit in X, and X is not complete.

- (c) First, we show that X is dense in c_0. If $x = (x_n) \in c_0$, then given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n| < \epsilon$ for all $n > N$. It follows that if
$$x^{(N)} = (x_1, \dots, x_N, 0, 0, \dots) \in X,$$
then $\left\|x - x^{(N)}\right\| < \epsilon$, so X is a dense subspace of c_0.

- Next, we prove that c_0 is complete. Suppose that $\left(x^{(k)}\right)$ is a Cauchy sequence in c_0, where
$$x^{(k)} = \left(x_n^{(k)}\right)_{n=1}^{\infty}.$$
Since
$$\left|x_n^{(k)} - x_n^{(\ell)}\right| \le \left\|x^{(k)} - x^{(\ell)}\right\|,$$
the sequence $(x_n^{(k)})_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$ for each $n \in \mathbb{N}$, so by the completeness of $\mathbb{R}$, there is $x_n \in \mathbb{R}$ such that
$$x_n^{(k)} \to x_n \qquad \text{as } k \to \infty.$$

- Let $x = (x_n)$ and let $\epsilon > 0$ be given. Since $\left(x^{(k)}\right)$ is Cauchy in c_0, there exists $K_\epsilon \in \mathbb{N}$ such that
$$\left|x_n^{(k)} - x_n^{(\ell)}\right| < \epsilon \qquad \text{for every } n \in \mathbb{N} \text{ and all } k, \ell \ge K_\epsilon.$$
Taking the limit of this inequality as $\ell \to \infty$, we get that
$$|x_n^{(k)} - x_n| \le \epsilon \qquad \text{for every } n \in \mathbb{N} \text{ and all } k \ge K_\epsilon.$$
It follows that that
$$\|x^{(k)} - x\| = \sup_{n \in \mathbb{N}} |x_n^{(k)} - x_n| \le \epsilon \qquad \text{for } k \ge K_\epsilon,$$
which shows that $\|x^{(k)} - x\| \to 0$ as $k \to \infty$.

- Finally, we show that $x \in c_0$. Let $\epsilon > 0$ be given. Then there exists $k_\epsilon \in \mathbb{N}$ such that
$$\|x - x^{(k_\epsilon)}\| < \frac{\epsilon}{2},$$
and since $x^{(k_\epsilon)} \in c_0$, there exists $N_\epsilon \in \mathbb{N}$ such that
$$\left|x_n^{(k_\epsilon)}\right| < \frac{\epsilon}{2} \qquad \text{for } n > N_\epsilon.$$
It follows that
$$|x_n| \le \left|x_n - x_n^{(k_\epsilon)}\right| + \left|x_n^{(k_\epsilon)}\right| < \epsilon \qquad \text{for } n > N_\epsilon,$$
which shows that $x \in c_0$ and c_0 is complete.