Problem Set 2: Math 201B
Due: Friday, January 14

1. If $1 \leq p < \infty$, show that the trigonometric polynomials are dense in $L^p(\mathbb{T})$.

2. For fixed $z \in \mathbb{C}$, let $J_n(z)$ denote the nth Fourier coefficient of the function $e^{iz\sin x}$, meaning that

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin x}e^{-inx}\,dx \qquad \text{for } n \in \mathbb{Z}.$$

(a) What is $J_n(0)$? Show that $J_{-n}(z) = (-1)^n J_n(z)$.
(b) Derive the recurrence relations

$$\frac{2n}{z}J_n(z) = J_{n-1}(z) + J_{n+1}(z), \qquad 2J_n'(z) = J_{n-1}(z) - J_{n+1}(z)$$

where the prime denotes a derivative with respect to z.
(c) Deduce from (b) that $J_n(z)$ is a solution of Bessel's equation

$$z^2 J_n'' + zJ_n' + \left(z^2 - n^2\right) J_n = 0.$$

3. A family of (not necessarily positive) functions $\{\phi_n \in L^1(\mathbb{T}) : n \in \mathbb{N}\}$ is an approximate identity if:

$$\int \phi_n\,dx = 1 \qquad \text{for every } n \in \mathbb{N};$$

$$\int |\phi_n|\,dx \leq M \qquad \text{for some constant } M \text{ and all } n \in \mathbb{N};$$

$$\lim_{n\to\infty}\int_{\delta<|x|<\pi} |\phi_n|\,dx = 0 \qquad \text{for every } \delta > 0.$$

If $f \in L^1(\mathbb{T})$, show that $\phi_n * f \to f$ in $L^1(\mathbb{T})$ as $n \to \infty$.

4. (a) Let $\{a_n : n \geq 0\}$ be a sequence of non-negative real numbers such that $a_n \to 0$ as $n \to \infty$ and

$$a_{n+1} - 2a_n + a_{n-1} \geq 0.$$

Show that the series

$$\sum_1^\infty n\left(a_{n+1} - 2a_n + a_{n-1}\right)$$

converges to a_0. HINT. $\sum(a_{n+1}-a_n)$ is a convergent, decreasing telescoping series.

(b) For $N \geq 0$, let $K_N \geq 0$ denote the Fejér kernel

$$K_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{inx}.$$

Show that the series

$$f(x) = \sum_{n=1}^{\infty} n\,(a_{n+1} - 2a_n + a_{n-1})\,K_{n-1}(x)$$

converges in $L^1(\mathbb{T})$ to a non-negative function $f \in L^1(\mathbb{T})$ whose Fourier coefficients are $a_{|n|}$ *i.e.*

$$f(x) \sim \sum_{n\in\mathbb{Z}} a_{|n|}e^{inx}.$$

(c) Show that there is a function $f \in L^1(\mathbb{T})$ such that

$$f(x) \sim \sum_{|n|\geq 2} \frac{1}{\log|n|}e^{inx}.$$

(d) Suppose that $f \in L^1(\mathbb{T})$ has imaginary Fourier coefficients $\{ib_n : n \in \mathbb{Z}\}$ such that $b_n \geq 0$ for $n \geq 0$ and $b_{-n} = -b_n$. Show that

$$\sum_{n=1}^{\infty} \frac{b_n}{n}$$

converges. HINT. The integral

$$F(x) = \int_0^x f(t)\,dt$$

is a continuous function (in fact, absolutely continuous) with Fourier coefficients

$$\frac{1}{2\pi}\int F(x)e^{-inx}\,dx = \frac{b_n}{in} \qquad \text{for } n \neq 0.$$

Use the fact that $K_N * F(0)$ converges to $F(0)$ since $\{K_N\}$ is an approximate identity.

(e) Show that there is no function $f \in L^1(\mathbb{T})$ such that

$$f(x) \sim \sum_{|n|\geq 2} \frac{i\,\mathrm{sgn}\,n}{\log|n|}e^{inx}.$$

(Here, $\mathrm{sgn}\,n$ is equal to 1 if $n > 0$ and -1 if $n < 0$.)