Methods of Applied Mathematics

Math 207A, Fall 2018

Midterm: Solutions

1 [40%] An SIR model for the spread of a disease in a population is given by the following equations:

$$\begin{aligned} S_t &= a\,(I + R + S) - aS - bSI, \\ I_t &= bSI - (a + c)I, \\ R_t &= cI - aR, \end{aligned}$$

where $S(t)$ is the number of susceptible individuals, $I(t)$ is the number of infected individuals, and $R(t)$ is the number of recovered individuals at time t. The positive parameters $a, b, c > 0$ have the following interpretations: a is the birth rate, which is assumed to equal the death rate; b is the transmission likelihood when a susceptible individual comes into contact with an infected individual; and c is the recovery rate. Recovered individuals are immune to the disease.

(a) Show that $S(t) + I(t) + R(t) = N$ is constant (where $N > 0$ is the total population).

(b) Introduce dimensionless variables $T = at$, $x = S/N$, $y = I/N$, $z = R/N$, and show that $x(T)$, $y(T)$ satisfy

$$x_T = 1 - x - \beta xy, \qquad y_T = \beta xy - (1 + \gamma)y$$

for suitable dimensionless parameters $\beta, \gamma > 0$.

(c) Find all fixed points with $x, y \geq 0$ and (where possible) use linearization to determine their stability. Consider all parameter values $\beta, \gamma > 0$. What do your results say in terms of modeling a disease?

Solution

- (a) It follows from the ODEs that $(S + I + R)_t = S_t + I_t + R_t = 0$, so $S + I + R = \text{constant}$.

- (b) Writing $\partial_t = a\partial_T$ and transforming to dimensionless variables, we get that

$$\begin{aligned} x_T &= (x + y + z) - x - \beta xy, \\ y_T &= \beta xy - (1 + \gamma)y, \\ z_t &= \gamma y - z, \end{aligned}$$

where

$$\beta = \frac{bN}{a}, \qquad \gamma = \frac{c}{a}.$$

Note that $[a] = [c] = 1/T$ and $[b] = 1/TP$ where T denotes a dimension of time and P denotes a dimension of population, so β, γ are dimensionless.

- From (a), we have $x + y + z = 1$, so elimination of z from the equation for x gives the stated equations for (x, y).

- (c) The equilibria satisfy

$$1 - x - \beta xy = 0, \qquad (\beta x - (1 + \gamma))\, y = 0.$$

From the second equation, either $y = 0$ when $x = 1$ from the first equation, or $x = \bar{x}$ when $y = \bar{y}$ where

$$\bar{x} = \frac{1 + \gamma}{\beta}, \qquad \bar{y} = \frac{1}{1 + \gamma} - \frac{1}{\beta}.$$

This solution for $\bar{y}$ is only nonnegative when $\beta \geq 1 + \gamma$, in which case $0 \leq \bar{x}, \bar{y}, \bar{z} \leq 1$, where $\bar{z} = 1 - \bar{x} - \bar{y}$.

- The Jacobian matrix of the system is

$$Df(x, y) = \begin{pmatrix} -1 - \beta y & -\beta x \\ \beta y & \beta x - (1 + \gamma) \end{pmatrix}.$$

- For the equilibrium $(x, y) = (1, 0)$, we have

$$Df(1, 0) = \begin{pmatrix} -1 & -\beta \\ 0 & \beta - (1 + \gamma) \end{pmatrix},$$

with eigenvalues $\lambda_1 = -1$, $\lambda_2 = \beta - (1 + \gamma)$. Hence, $(1, 0)$ is an asymptotically stable node if $\beta < 1 + \gamma$ and an unstable saddle point if $\beta > 1 + \gamma$. The equilibrium is nonhyperbolic if $\beta = 1 + \gamma$, and we can't conclude its stability from the linearization.

- For the equilibrium $(x, y) = (\bar{x}, \bar{y})$, we have

$$Df(\bar{x}, \bar{y}) = \begin{pmatrix} -\beta/(1 + \gamma) & -(1 + \gamma) \\ \beta/(1 + \gamma) - 1 & 0 \end{pmatrix},$$

with eigenvalues

$$\lambda = \frac{1}{2}\left[-\frac{\beta}{1+\gamma} \pm \sqrt{\left(\frac{\beta}{1+\gamma}\right)^2 - 4\left(\beta - (1+\gamma)\right)}\right].$$

If $\beta > 1+\gamma$, then both of these eigenvalues have negative real part, and the equilibrium is an asymptotically stable node or spiral point. If $\beta = 1+\gamma$, then the equilibrium coincides with $(1,0)$.

- In dimensional terms, it follows that the equilibrium $(S, I, R) = (N, 0, 0)$ with no disease present is asymptotically stable when $bN < a+c$, meaning that the transmission rate is less that the sum of the death rate and the recovery rate. If $bN > a+c$, then this equilibrium loses stability and the new stable state $(S, I, R) = (N\bar{x}, N\bar{y}, N\bar{z})$ is one in which a nonzero fraction of the population is infected.

Remark. As β increases through $1+\gamma$, the two equilibria cross and exchange stability. This is an example of a transcritical bifurcation.

2 [30%] Duffing's equation is

$$\ddot{x} + \delta\dot{x} + x - x^3 = 0.$$

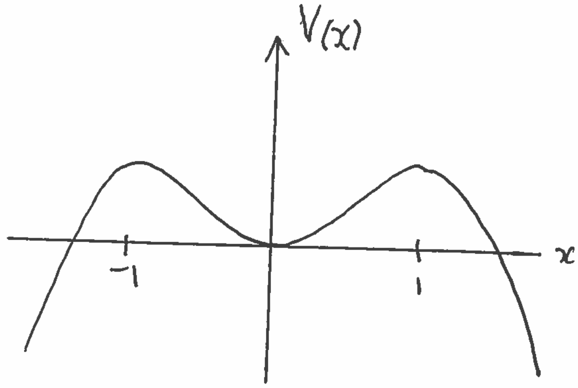
(a) Sketch the $(x, \dot{x})$-phase plane for $\delta = 0$. Classify the equilibria and identify any homoclinic or heteroclinic orbits.

(b) Sketch the $(x, \dot{x})$-phase plane for $0 < \delta \ll 1$. Classify the equilibria and indicate the points in the phase plane whose ω-limit set consists of the point $(x, \dot{x}) = (0, 0)$.

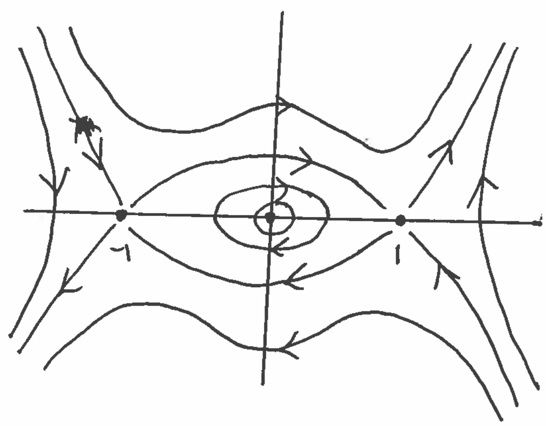
Solution

- (a) For $\delta = 0$, the system is a conservative system $\ddot{x} + V'(x) = 0$ with potential
$$V(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4.$$
There are three equilibria: $(0, 0)$ is a nonlinear center; and $(\pm 1, 0)$ are saddle points. There are two heteroclinic orbits, one connecting $(-1, 0)$ to $(1, 0)$, the other connecting $(1, 0)$ to $(-1, 0)$. The phase plane is sketched on the next page.

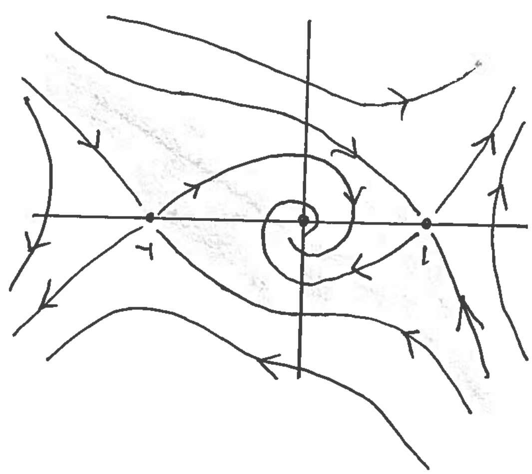
- (b) When small damping is included, $(0, 0)$ becomes an asymptotically stable spiral point, and $(\pm 1, 0)$ remain saddle points. The basin of attraction of $(0, 0)$ is the shaded region enclosed by the two stable manifolds of the saddle points.



(a)



(b)



3 [30%] Consider the system

$$\dot{x} = y, \qquad \dot{y} = -(x^2 + y^2 - 4)y - x^3.$$

(a) Let

$$E(x, y) = \frac{1}{4}x^4 + \frac{1}{2}y^2.$$

Derive an equation for $\dot{E}\,(x(t), y(t))$, and show that there exist constants $0 < a < b$ such that $a \leq E(x, y) \leq b$ is a trapping region for the flow.
(b) Show that the system has a limit cycle in the region $a \leq E(x, y) \leq b$.

Solution

- (a) We compute that

$$\dot{E} = x^3\dot{x} + y\dot{y} = -y^2\left(x^2 + y^2 - 4\right).$$

  It follows that E is decreasing on trajectories if $x^2 + y^2 \geq 4$ and increasing on trajectories if $x^2 + y^2 \leq 4$.

- For $c \geq 0$, the level set $E(x, y) = c$ is compact, so the continuous function $x^2 + y^2$ attains its maximum value $M(c)$ and minimum value $m(c)$ on the level set. Moreover, $M(c) \to 0$ as $c \to 0^+$ and $m(c) \to \infty$ as $c \to \infty$. Choose $0 < a < b$ such that $M(a) \leq 4$ and $m(b) \geq 4$. Then the compact set $a \leq E(x, y) \leq b$ is invariant since $x^2 + y^2 \leq 4$ on $E(x, y) = a$, so E is increasing, and $x^2 + y^2 \geq 4$ on $E(x, y) = b$, so E is decreasing.

- (b) The only equilibrium of the system is $(x, y) = (0, 0)$, so the invariant region $a \leq E(x, y) \leq b$ doesn't contain any equilibria. The Poincaré-Bendixson theorem implies that the ω-limit set of any orbit starting in the region is a periodic solution, so the region contains a periodic solution (which is, in fact, a limit cycle).

Remark. Using the method of Lagrange multipliers, one can show that the optimal values for enclosing the circle $x^2 + y^2 = 4$ between the curves $E(x, y) = a$ and $E(x, y) = b$ are $a = 7/4$ and $b = 4$. Alternatively, note that

$$x^2 + y^2 = 2c + x^2 - \frac{1}{2}x^4, \quad 0 \leq x^2 \leq 2\sqrt{c} \qquad \text{on } E(x, y) = c.$$

For $c \geq 1$, we have $M(c) = 2c + 1/2$, attained at $x^2 = 1$, and $m(c) = 2\sqrt{c}$, attained at $x^2 = 2\sqrt{c}$, so $M(c) = 4$ when $c = 7/4$ and $m(c) = 4$ when $c = 4$.