Sample Final Questions
Math 207B, Winter 2012
Brief Solutions

1. Find an explicit expression for the Green's function for the problem

$$
\begin{aligned}
&-u'' + u = f(x), \qquad 0 < x < 1\\
&u'(0) = 0, \qquad u'(1) = 0.
\end{aligned}
$$

Write down the Green's function representation of the solution for $u(x)$.

Solution

- Homogeneous solutions that satisfy the BCs at the left and right end-points are
$$u_1(x) = \cosh x, \qquad u_2(x) = \cosh(1-x)$$
with Wronskian $-\sinh 1$, so the Green's function is
$$G(x,\xi) = \begin{cases} \cosh x \cosh(1-\xi)/\sinh 1 & 0 \le x \le \xi, \\ \cosh \xi \cosh(1-x)/\sinh 1 & \xi \le x \le 1. \end{cases}$$

- The Green's function representation of the solution is
$$u(x) = \int_0^1 G(x,\xi) f(\xi)\, d\xi.$$

2. Use separation of variables and Fourier series to solve the following IBVP for the Schrödinger equation for the complex-valued function $\psi(x,t)$

$$\begin{aligned} &i\psi_t = -\psi_{xx}, && 0 < x < 1 \\ &\psi(0,t) = 0, && \psi(1,t) = 0, \\ &\psi(x,0) = f(x) \end{aligned}$$

where $f \in L^2(0,1)$ is given initial data. Show from your solution that

$$\int_0^1 |\psi(x,t)|^2\,dx = \int_0^1 |f(x)|^2\,dx \qquad \text{for all } t \in \mathbb{R}.$$

Solution

- The solution is

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n e^{-in^2\pi^2 t} \sin(n\pi x)$$

  where

$$c_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx.$$

- By Parseval's theorem, and the fact that $|e^{-i\theta}| = 1$, we have for any $t \in \mathbb{R}$ that

$$\begin{aligned} \int_0^1 |\psi(x,t)|^2\,dx &= \frac{1}{2}\sum_{n=1}^{\infty} \left| c_n e^{-in^2\pi^2 t} \right|^2 \\ &= \frac{1}{2}\sum_{n=1}^{\infty} |c_n|^2 \\ &= \int_0^1 |f(x)|^2\,dx. \end{aligned}$$

3. After non-dimensionalization, the displacement $u(x)$ of a non-uniform string, with density $\rho(x)$, fixed at each end and vibrating with frequency ω satisfies the EVP

$$\begin{aligned} -u'' &= \lambda\rho(x)u, \qquad 0 < x < 1, \\ u(0) &= 0, \qquad u(1) = 0 \end{aligned}$$

where $\lambda = \omega^2$. The fundamental frequency of the string is $\omega_1 = \sqrt{\lambda_1}$, where $\lambda = \lambda_1$ is the smallest eigenvalue. If $m \le \rho(x) \le M$ where m, M are positive constants, show that

$$\frac{\pi}{\sqrt{M}} \le \omega_1 \le \frac{\pi}{\sqrt{m}}.$$

Does this result make sense physically?

Solution

- The Rayleigh quotient for the minimum eigenvalue is

$$\lambda_1 = \min_{u\neq 0} \frac{\int_0^1 u'(x)^2\,dx}{\int_0^1 \rho(x)u(x)^2\,dx}.$$

- The Rayleigh quotient for the minimum eigenvalue μ_1 of the problem with constant density ρ_0

$$\begin{aligned} -u'' &= \mu\rho_0 u, \qquad 0 < x < 1, \\ u(0) &= 0, \qquad u(1) = 0 \end{aligned}$$

  is

$$\mu_1 = \min_{u\neq 0} \frac{\int_0^1 u'(x)^2\,dx}{\int_0^1 \rho_0 u(x)^2\,dx}.$$

  In this case, we have an explicit solution for the minimum eigenvalue

$$\mu_1 = \frac{\pi^2}{\rho_0}$$

  with eigenfunction $\sin(\pi x)$.

- If $\rho(x) \geq m$ for all $x \in [0,1]$ then (taking $\rho_0 = m$)

  $$\int_0^1 \rho(x)u(x)^2\,dx \geq \int_0^1 mu(x)^2\,dx$$

  for every function $u(x)$, so

  $$\frac{\int_0^1 u'(x)^2\,dx}{\int_0^1 \rho(x)u(x)^2\,dx} \leq \frac{\int_0^1 u'(x)^2\,dx}{\int_0^1 mu(x)^2\,dx}.$$

  It follows that $\lambda_1 \leq \mu_1$, or

  $$\omega_1 \leq \frac{\pi}{\sqrt{m}}$$

- If $\rho(x) \leq M$ for all $x \in [0,1]$ then (taking $\rho_0 = M$)

  $$\int_0^1 \rho(x)u(x)^2\,dx \leq \int_0^1 Mu(x)^2\,dx$$

  for every function $u(x)$, so

  $$\frac{\int_0^1 u'(x)^2\,dx}{\int_0^1 \rho(x)u(x)^2\,dx} \geq \frac{\int_0^1 u'(x)^2\,dx}{\int_0^1 Mu(x)^2\,dx}.$$

  It follows that $\lambda_1 \geq \mu_1$, or

  $$\omega_1 \geq \frac{\pi}{\sqrt{M}}$$

- The result states that the fundamental frequency of a nonuniform string is greater than that of a heavier uniform string and less than that of a lighter uniform string, which is what one would expect physically.

4. Consider the Volterra integral operator $K : L^2(0,1) \to L^2(0,1)$ defined by

$$Ku(x) = \int_0^x u(y)\, dy, \qquad 0 < x < 1$$

Show that the integral equation $Ku = \lambda u$ has no nonzero solutions for any $\lambda \in \mathbb{C}$, meaning that K has no eigenvalues. Why doesn't this contradict the spectral theorem for compact (or Hilbert-Schmidt) self-adjoint operators?

Solution

- If $\lambda = 0$, then $Ku = 0$ and

  $$u = (Ku)' = 0,$$

  so 0 is not an eigenvalue of K.

- If $\lambda \neq 0$, then differentiating the equation $Ku = \lambda u$, and also setting $x = 0$ in the integral equation, we get

  $$\lambda u' = u, \qquad u(0) = 0.$$

  The general solution of the ODE is

  $$u(x) = ce^{x/\lambda}.$$

  The IC implies that $c = 0$, so $u = 0$ and λ is not an eigenvalue of K.

- The operator K is Hilbert-Schmidt, but it is not self-adjoint on $L^2(0,1)$. In fact, its adjoint is

  $$(K^* u)(x) = \int_x^1 u(y)\, dy$$

  (Moral: The spectral theory of non-self-adjoint operators is not nearly as nice as the theory for self-adjoint operators.)

5. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded region, and define an operator L by

$$Lu = -\nabla \cdot (p\nabla u) + qu$$

where p, q are smooth functions on $\bar{\Omega}$. Show that

$$\int_\Omega uLv\,dx = \int_\Omega vLu\,dx$$

for all functions $u, v : \Omega \to \mathbb{R}$ that vanish on the boundary $\partial\Omega$, meaning that L with Dirichlet BCs is formally self-adjoint.

Solution

- We have the identity

$$u\nabla \cdot (p\nabla v) - v\nabla \cdot (p\nabla u) = \nabla \cdot (pu\nabla v - pv\nabla u)\,.$$

  To show this, we compute in Cartesian components (using the summation convention) that

$$\begin{aligned}\nabla \cdot (pu\nabla v - pu\nabla v) &= \frac{\partial}{\partial x_i}\left(pu\frac{\partial v}{\partial x_i} - pv\frac{\partial u}{\partial x_i}\right)\\ &= u\frac{\partial}{\partial x_i}\left(p\frac{\partial v}{\partial x_i}\right) + p\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\\ &\qquad - v\frac{\partial}{\partial x_i}\left(p\frac{\partial u}{\partial x_i}\right) - p\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}\\ &= u\frac{\partial}{\partial x_i}\left(p\frac{\partial v}{\partial x_i}\right) - u\frac{\partial}{\partial x_i}\left(p\frac{\partial v}{\partial x_i}\right)\\ &= u\nabla \cdot (p\nabla v) - v\nabla \cdot (p\nabla u)\,.\end{aligned}$$

- Using this identity and the divergence theorem, we get

$$\begin{aligned}\int_\Omega (uLv - vLu)\ dx &= \int_\Omega \{-u\nabla \cdot (p\nabla v) + quv + v\nabla \cdot (p\nabla u) - quv\}\ dx\\ &= -\int_\Omega \{u\nabla \cdot (p\nabla v) - v\nabla \cdot (p\nabla u)\}\ dx\\ &= -\int_\Omega \nabla \cdot (pu\nabla v - pu\nabla v)\ dx\\ &= -\int_{\partial\Omega}\left(pu\frac{\partial v}{\partial n} - pv\frac{\partial u}{\partial n}\right)\ dS.\end{aligned}$$

The integral over the boundary vanishes since $u, v = 0$ on $\partial\Omega$, so

$$\int_\Omega uLv\,dx = \int_\Omega vLu\,dx$$

- This is a multi-dimensional analog of the corresponding self-adjointness identity for the one-dimensional Sturm-Liouville operator

  $$Lu = -(pu')' + qu$$

  with Dirichlet BCs.

6. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded region, Consider the Neumann BVP

$$\begin{aligned} -\Delta u &= f(x) \qquad x \in \Omega, \\ \frac{\partial u}{\partial n} &= g(x) \qquad x \in \partial\Omega. \end{aligned}$$

(a) Show that a solution can only exist if

$$\int_\Omega f dx + \int_{\partial\Omega} g dS = 0$$

Give a physical interpretation of this result in terms of heat flow.
(b) If a solution exists, show that it is unique up to an arbitrary additive constant.

Solution

- (a) Assume there is a solution u. Then, using the divergence theorem, we get

$$\int_\Omega f\, dx = -\int_\Omega \Delta u\, dx = -\int_{\partial\Omega} \frac{\partial u}{\partial n}\, dS = -\int_{\partial\Omega} g dS$$

  which gives the result.

- The problem describes the equilibrium temperature of a body with heat source density f and prescribed heat flux g into the body through its boundary. An equilibrium solution is only possible if the net rate at which internal sources generate heat ($\int_\Omega f\, dx$) is equal to the heat flux out of the body ($-\int_{\partial\Omega} g\, dS$).

- (b) Suppose u_1, u_2 are two solutions, and let $v = u_1 - u_2$. Then, by linearity,

$$\begin{aligned} \Delta v &= 0 \qquad x \in \Omega, \\ \frac{\partial v}{\partial n} &= 0 \qquad x \in \partial\Omega. \end{aligned}$$

- According to Green's first identity

$$\int_\Omega \left(v\Delta v + |\nabla v|^2\right)\, dx = \int_\Omega \nabla \cdot (v\nabla v)\, dx = \int_{\partial\Omega} v\frac{\partial v}{\partial n}\, dS.$$

Using the equations for v in this identity, we get that

$$\int_{\Omega} |\nabla v|^2 \, dx = 0.$$

It follows that $\nabla v = 0$ in Ω, meaning that v is constant, so any two solutions are equal up to a constant.