Advanced Calculus
Math 25, Fall 2015
Final: Solutions

1. [20 pts] Say if the following statements are true or false. If false, give a counter-example, if true give a brief explanation why (a complete proof is not required).

(a) If (a_n) is a sequence such that for every $k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $a_j = a_{j+1} = \cdots = a_{j+k}$ (meaning that the sequence contains arbitrarily long strings of repeated terms), then (a_n) converges.

(b) If the series $\sum a_n$ is conditionally convergent, then the series $\sum \sqrt{n} a_n$ diverges.

(c) If $A \subset \mathbb{R}$ and every $a \in A$ is an interior point of A, then A is open.

(d) If $A \subset \mathbb{R}$ and every $a \in A$ is an isolated point of A, then A is closed.

Solution.

- (a) False. For example, the sequence $(1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots)$ with n successive integers n does not converge.
- (b) False. For example, if $a_n = (-1)^{n+1}/n$, then the alternating harmonic series $\sum(-1)^{n+1}/n$ is conditionally convergent and $\sum(-1)^{n+1}/\sqrt{n}$ converges by the alternating series test.
- (c) True. This follows immediately from the definitions. If $a \in A$ is an interior point, then there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subset A$, so A is open if (and only if) every $a \in A$ is an interior point.
- (d) False. For example, if $A = \{1/n : n \in \mathbb{N}\}$, then every point of A is an isolated point, but A is not closed since $0 \notin A$ is a limit point of A.

2. [20 pts] Prove by induction that

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$$

for every natural number $n \in \mathbb{N}$.

HINT. You can use the fact that the sum of the first n natural numbers is given by $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$.

Solution.

- The result is true for $n=1$, since $1^2 = 1^3$.

- Assume the result is true for some $n \in \mathbb{N}$. Then, using the induction hypothesis and the sum given in the hint, we get that

$$\begin{aligned}
1^3+2^3+3+\cdots+n^3+(n+1)^3 &= (1+2+3+\cdots+n)^2+(n+1)^3 \\
&= \left[\frac{1}{2}n(n+1)\right]^2+(n+1)^3 \\
&= (n+1)^2\left[\frac{1}{4}n^2+n+1\right] \\
&= \frac{1}{4}(n+1)^2(n+2)^2 \\
&= (1+2+3+\cdots+n+n+1)^2,
\end{aligned}$$

  so the result is true for $n+1$. It follows by induction that the result holds for every $n \in \mathbb{N}$.

- Alternatively, you could prove by induction that

$$1^3+2^3+\cdots+n^3 = \frac{1}{4}n^2(n+1)^2$$

  and observe that the result follows from the sum in the hint.

3. [20 pts] Suppose that the sets $A, B \subset \mathbb{R}$ are bounded from above. Let

$$A + B = \{x \in \mathbb{R} : x = a + b \text{ for some } a \in A \text{ and } b \in B\}.$$

Prove that $\sup(A + B) = \sup A + \sup B$.

Solution.

- Let $P = \sup A$, $Q = \sup B$, and $M = P + Q$. Since P is an upper bound of A and Q is an upper bound of B, we have $a + b \leq P + Q$ for every $a \in A$ and $b \in B$, so M is an upper bound of $A + B$.

- Suppose that $M' < M$ and let $\epsilon = M - M' > 0$. If $P' = P - \epsilon/2$ and $Q' = Q - \epsilon/2$, then $P' + Q' = M'$. Since P is a least upper bound of A and $P' < P$, there exists $a \in A$ such that $a > P'$; similarly, there exists $b \in B$ such that $b > Q'$. It follows that $a + b > M'$, so M' is not an upper bound of $A + B$, which proves that M is a least upper bound and $\sup(A + B) = \sup A + \sup B$.

4. [20 pts] (a) State the definition of the convergence of a sequence (a_n) of real numbers to a limit L.

(b) Suppose that $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$. Prove from the definition that $\lim_{n\to\infty} \sqrt{a_n} = 0$.

Solution.

- (a) $a_n \to L$ as $n \to \infty$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|a_n - L| < \epsilon$.
- (b) Let $\epsilon > 0$. Since $a_n \to 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies that $0 \leq a_n < \epsilon^2$. It follows that $n > N$ implies that $\sqrt{a_n} < \epsilon$, which proves that $\sqrt{a_n} \to 0$ as $n \to \infty$.

5. [20 pts] Let (a_n) be a bounded sequence of real numbers.
(a) State the definition of $\limsup_{n\to\infty} a_n$.
(b) Prove that there is a subsequence (a_{n_k}) of (a_n) such that

$$\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n.$$

(c) Prove that if (a_{n_k}) is any convergent subsequence of (a_n), then

$$\lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n.$$

Solution.

- (a)

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n, \qquad b_n = \sup\{a_k : k \geq n\}.$$

- (b) Let $L = \limsup a_n$, so $b_n \downarrow L$. For each $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that $n > N_k$ implies that

$$L \leq b_n < L + \frac{1}{k+1}.$$

We construct a subsequence (a_{n_k}) recursively as follows. First, choose any $n_1 \in \mathbb{N}$. Then, given n_k for $k \in \mathbb{N}$, choose some $n > \max\{n_k, N_k\}$. By the definition of the supremum that defines b_n, there exists $n_{k+1} \geq n$ such that

$$b_n - \frac{1}{k+1} \leq a_{n_{k+1}} \leq b_n,$$

It follows that $n_{k+1} > n_k$ and

$$L - \frac{1}{k+1} < a_{n_{k+1}} < L + \frac{1}{k+1},$$

or $|a_{n_k} - L| < 1/k$, for every $k \geq 2$. The sandwich theorem then implies that $a_{n_k} \to L$ as $k \to \infty$.

- (c) Suppose that (a_{n_k}) is a convergent subsequence of (a_n). For every $k \in \mathbb{N}$, we have $a_{n_k} \leq b_{n_k}$. Taking the limit of this inequality as $k \to \infty$, then using the monotonicity property of limits and the fact that every subsequence of (b_n) converges to the same limit as (b_n), we get that

$$\lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n.$$

6. [20 pts] Determine the convergence of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n!}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{4^n-3}{n^4+3}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n} - \frac{n}{n+1}\right)$

Solution.

- (a) If $n \geq 16$, then $2/\sqrt{n} \leq 1/2$, and it follows that

$$\frac{2^n}{\sqrt{n!}} = \frac{2}{\sqrt{1}} \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{3}} \cdots \frac{2}{\sqrt{n}} \leq \frac{C}{2^n}$$

  for a suitable constant C that is independent of n. So the series converges absolutely by comparison with a convergent geometric series.

- (b) For $n \geq 2$, we have $n - 1 \geq n/2$ and

$$\frac{n-1}{n^2+1} > \frac{n/2}{n^2+n^2} = \frac{1}{4n}.$$

  It follows that the series diverges to ∞ by comparison with the divergent harmonic series.

- (c) The series diverges because $4^n/n^4 \to \infty$ as $n \to \infty$, so its terms diverge to ∞. To prove this limit, note that for $n \geq 6$ we have

$$4^n = (1+3)^n > \binom{n}{4} 3^{n-4} = \frac{n(n-1)(n-2)(n-3)3^n}{4!3^4} > \frac{n^4 3^n}{4!3^4 2^3},$$

  where we retain only one term in the binomial expansion of $(1+3)^n$, and $3^n \to \infty$ as $n \to \infty$.

- (d) This is a telescoping series of negative terms, and

$$\sum_{n=1}^{N} \left|\frac{n-1}{n} - \frac{n}{n+1}\right| = \sum_{n=1}^{N} \left(\frac{n}{n+1} - \frac{n-1}{n}\right) = \frac{N}{N+1} - 0 \to 1$$

  as $N \to \infty$, so the series converges absolutely (to -1).

7. [20 pts] (a) Define an open set $G \subset \mathbb{R}$.

(b) If $\{G_1, G_2, \ldots, G_n\}$ is a finite collection of open sets $G_i \subset \mathbb{R}$, prove that $G = \bigcap_{i=1}^{n} G_i$ is open

(c) If $\{G_i : i \in \mathbb{N}\}$ is an infinite collection of open sets $G_i \subset \mathbb{R}$, give an example to show that that $\bigcap_{i=1}^{\infty} G_i$ need not be open.

Solution.

- (a) A set $G \subset \mathbb{R}$ is open if for every $x \in G$ there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset G$.

- (b) Suppose that $x \in \bigcap_{i=1}^{n} G_i$. Then $x \in G_i$ for every $1 \leq i \leq n$. Since G_i is open, there exists $\delta_i > 0$ such that $(x - \delta_i, x + \delta_i) \subset G_i$. If

$$\delta = \min\{\delta_i : 1 \leq i \leq n\},$$

  then $\delta > 0$ and $(x-\delta, x+\delta) \subset G_i$ for every i, so $(x-\delta, x+\delta) \subset \bigcap_{i=1}^{n} G_i$, which proves that $\bigcap_{i=1}^{n} G_i$ is open.

- (c) Let

$$G_i = \left(0, 1 + \frac{1}{i}\right)$$

  Then G_i is open but

$$\bigcap_{i=1}^{\infty} G_i = (0, 1]$$

  is not open.