Analysis Preliminary Exam Workshop:
Hilbert Spaces

1. Hilbert spaces

A Hilbert space $\mathcal{H}$ is a complete real or complex inner product space. Consider complex Hilbert spaces for definiteness. If $(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is the inner product, then

$$\|x\| = \sqrt{(x,x)}, \qquad \|\cdot\| : \mathcal{H} \to [0,\infty)$$

defines a norm on $\mathcal{H}$, and $\mathcal{H}$ is complete with respect to this norm.

Examples: $\mathbb{R}^n$ (or $\mathbb{C}^n$) with the standard Euclidean (or Hermitian) inner product; $\ell^2(\mathbb{N})$; $L^2(\mathbb{R})$; $L^2(\mathbb{T})$.

Theorem 1 (Cauchy-Schwarz). *If $\mathcal{H}$ is a Hilbert space and $x, y \in \mathcal{H}$, then $|(x,y)| \le \|x\| \, \|y\|$.*

2. Geometry

Two vectors $x, y \in \mathcal{H}$ are orthogonal, written $x \perp y$, if $(x,y) = 0$. If $X \subset \mathcal{H}$, then

$$X^\perp = \{y \in \mathcal{H} : (x,y) = 0 \text{ for all } x \in X\}.$$

Theorem 2 (Orthogonal complements). *If $\mathcal{M}$ is a closed linear subspace of a Hilbert space $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $\mathcal{M}^\perp$ is also a closed linear subspace.*

A set of vectors $\{e_\alpha : \alpha \in I\}$ in $\mathcal{H}$ is orthonormal if

$$(e_\alpha, e_\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \ne \beta. \end{cases}$$

An orthonormal basis is a maximal orthonormal set.

THEOREM 3 (Basis). *Every Hilbert space $\mathcal{H}$ has an orthonormal basis $\{e_\alpha : \alpha \in I\}$. Every element $x \in \mathcal{H}$ has a norm-convergent expansion*

$$x = \sum_{\alpha \in I} x_\alpha e_\alpha, \qquad x_\alpha = (e_\alpha, x)$$

and $x \in \mathcal{H}$ if and only if

$$\sum_{\alpha \in I} |x_\alpha|^2 < \infty.$$

Moreover, any two orthonormal bases of $\mathcal{H}$ have the same cardinality.

Any two Hilbert spaces with bases of the same cardinality are isomorphic by a unitary transformation that maps an orthonormal basis of one space to an orthonormal basis of the other. A finite-dimensional complex Hilbert space is isomorphic to $\mathbb{C}^n$. An infinite-dimensional separable Hilbert space, with a countable orthonormal basis, is isomorphic to $\ell^2(\mathbb{N})$.

THEOREM 4 (Parseval). *If $\mathcal{H}$ is a Hilbert space with orthonormal basis $\{e_\alpha : \alpha \in I\}$ and $x, y \in \mathcal{H}$ have expansions*

$$x = \sum_{\alpha \in I} x_\alpha e_\alpha, \qquad y = \sum_{\alpha \in I} y_\alpha e_\alpha,$$

then

$$(x, y) = \sum_{\alpha \in I} \overline{x_\alpha} y_\alpha.$$

In particular,

$$\|x\|^2 = \sum_{\alpha \in I} |x_\alpha|^2.$$

3. Bounded linear operators

A linear operator $A : \mathcal{H} \to \mathcal{K}$ between Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ is bounded if its operator norm

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

is finite. We denote the Banach space of all such bounded linear operators by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, or $\mathcal{B}(\mathcal{H})$ if $\mathcal{H} = \mathcal{K}$.

Every bounded linear operator $A : \mathcal{H} \to \mathcal{K}$ has a (Hilbert space) adjoint $A^* : \mathcal{K} \to \mathcal{H}$ that is bounded and linear and uniquely defined by the condition

$$(x, Ay) = (A^*x, y) \qquad \text{for all } x \in \mathcal{K} \text{ and } y \in \mathcal{H}.$$

An operator $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint if $A = A^*$, meaning that

$$(x, Ay) = (Ax, y) \qquad \text{for all } x, y \in \mathcal{H},$$

unitary (or orthogonal in the real case) if $A^* = A^{-1}$, and normal if $A^*A = AA^*$.

Denote the kernel, or null space, of $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ by $\ker A$ (a closed subspace of $\mathcal{H}$) and the range of A by $\operatorname{ran} A$ (a not necessarily closed subspace of $\mathcal{K}$). The rank of an operator is the dimension of its range, and A has finite rank if $\operatorname{ran} A$ is finite-dimensional (in which case it is closed).

Theorem 5. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$\ker A = \operatorname{ran}(A^*)^\perp, \qquad \overline{\operatorname{ran}(A^*)} = (\ker A)^\perp.$$

An orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ is a self-adjoint bounded linear operator on $\mathcal{H}$ such that $P^2 = P$. It follows that $\|P\| = 1$, unless $P = 0$.

Theorem 6 (Projection). *If $\mathcal{M}$ is a closed linear subspace of a Hilbert space $\mathcal{H}$, then there is a orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ with range $\mathcal{M}$ and kernel $\mathcal{M}^\perp$. Conversely, if $P : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ where the closed subspaces $\mathcal{M}$ and $\mathcal{M}^\perp$ are the range and kernel of P, respectively.*

4. Compact operators

A subset K of a Hilbert space $\mathcal{H}$ is compact if every sequence in K has a bounded subsequence whose limit is in K. The set K is precompact if its closure is compact, meaning that every sequence in K has a convergent subsequence whose limit is in $\mathcal{H}$. A bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ between Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ is compact if it maps bounded sets in $\mathcal{H}$ to precompact sets in $\mathcal{K}$ (i.e. sets whose closure is compact).

Theorem 7. *An operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is compact if and only if there is a sequence of operators $T_n \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with finite rank such that*

$$\|T - T_n\| \to 0 \qquad \text{as } n \to \infty.$$

5. Spectrum

Suppose that $\mathcal{H}$ is a complex Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. The resolvent set $\rho(A) \subset \mathbb{C}$ of A is the set of $\lambda \in \mathbb{C}$ such that

$$(A - \lambda I) : \mathcal{H} \to \mathcal{H}$$

is one-to-one and onto.[1] The spectrum $\sigma(A)$ of A is the complement of $\rho(A)$ in $\mathbb{C}$. The resolvent set is open and the spectrum is compact (closed and bounded).

We classify the spectrum of A as follows:

- The point spectrum consists of the $\lambda \in \sigma(A)$ such that $A - \lambda I$ is not one-to-one (then λ is an eigenvalue of A and $x \neq 0$ such that $Ax = \lambda x$ an eigenvector);
- The continuous spectrum consists of the $\lambda \in \sigma(A)$ such that $A - \lambda I$ is one-to-one but not onto and $\operatorname{ran}(A - \lambda I)$ is a dense subspace of $\mathcal{H}$;
- The residual spectrum consists of $\lambda \in \sigma(A)$ such that $A - \lambda I$ is one-to-one but not onto and $\operatorname{ran}(A - \lambda I)$ is not dense in $\mathcal{H}$.

THEOREM 8. *If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then its spectrum $\sigma(A) \subset \mathbb{R}$ is real and the residual spectrum of A is empty*

6. Spectral theorem for compact self-adjoint operators

THEOREM 9 (Spectral theorem for compact self-adjoint operators). *Let $A \in \mathcal{B}(\mathcal{H})$ be a compact, self-adjoint operator on a Hilbert space $\mathcal{H}$. Then A has a finite or countably infinite sequence of real, non-zero eigenvalues (λ_n). If this sequence is countably infinite, then $\lambda_n \to 0$ as $n \to \infty$. Every eigenspace associated with a nonzero eigenvalue is finite dimensional. If 0 is an eigenvalue of A, the null space of A may be finite or infinite dimensional. Furthermore, $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of A.*

THEOREM 10 (Projection form of spectral theorem). *Suppose that $A \in \mathcal{B}(\mathcal{H})$ is a compact, self-adjoint operator on a Hilbert space $\mathcal{H}$ with distinct nonzero eigenvalues $(\lambda_n \to 0)$. Let P_n be the orthogonal projection onto*

[1] The open mapping theorem then implies that the inverse $(A - \lambda I)^{-1}$ is bounded.

the eigenspace associated with λ_n *and* P_0 *the projection onto the null space of* A*. Then* P_n *has finite rank,* $P_m P_n = 0$*, and*

$$A = \sum_n \lambda_n P_n,$$

where the series converges uniformly with respect to the operator norm. Moreover,

$$I = P_0 + \sum_n P_n,$$

where the series converges strongly i.e. $(P_0 + \sum_{n=1}^{N} P_n)x \to x$ *in norm as* $N \to \infty$ *for every* $x \in \mathcal{H}$*.*

6. Weak convergence

The dual space of every Hilbert space is isomorphic (real case) or anti-isomorphic (complex case) to the Hilbert space.

THEOREM 11 (Riesz representation). *Every bounded linear functional* $\omega : \mathcal{H} \to \mathbb{C}$ *has the form*

$$\omega(y) = (x, y)$$

for some $x \in \mathcal{H}$*, and* $\|\omega\| = \|x\|$*.*

A sequence (x_n) in $\mathcal{H}$ converges weakly to $x \in \mathcal{H}$, written $x_n \rightharpoonup x$, if

$$(x_n, y) \to (x, y) \qquad \text{for every } y \in \mathcal{H}.$$

THEOREM 12. *If* $x_n \rightharpoonup x$*, then* $\{x_n\}$ *is bounded and*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

THEOREM 13 (Banach-Alaoglu). *The closed unit ball in* $\mathcal{H}$ *is weakly compact.*