Bandlimited Fourier Transform
Analysis Prelim Worksop
Fall 2012

Problem 9. (a) For $f \in L^1(\mathbb{R})$ and $R > 0$, let

$$(S_R f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(\xi) e^{ix\xi}\, d\xi$$

where $\hat{f}$ is the Fourier transform of f, defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi}\, dx.$$

Show that

$$S_R f = K_R * f$$

where

$$K_R(x) = \frac{\sin Rx}{\pi x}$$

Show this result also holds for $f \in L^2(\mathbb{R})$.
(b) If $f \in L^2$, show that $S_R f \to f$ in L^2 as $R \to \infty$.

Solution. (a) For $f \in L^1(\mathbb{R})$,

$$\begin{aligned}(S_R f)(x) &= \frac{1}{2\pi} \int_{-R}^{R} \left[\int_{-\infty}^{\infty} f(y) e^{-iy\xi}\, dy \right] e^{ix\xi}\, d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \left[\int_{-R}^{R} e^{i(x-y)\xi}\, d\xi \right] dy \\ &= \int_{-\infty}^{\infty} K_R(x-y) f(y)\, dy,\end{aligned}$$

or $S_R f = K_R * f$, where

$$K_R(x) = \frac{1}{2\pi} \int_{-R}^{R} e^{ix\xi}\, d\xi = \frac{\sin Rx}{\pi x}.$$

Here, Fubini's theorem allows us to exchange the order of integration because

$$\int_{-R}^{R} \left[\int_{-\infty}^{\infty} \left| f(y) e^{i(x-y)\xi} \right| dy \right] d\xi = \int_{-R}^{R} \left[\int_{-\infty}^{\infty} |f(y)|\, dy \right] d\xi = 2R \|f\|_{L^1} < \infty.$$

An alternative derivation of this result is by the general convolution theorem. We have $K_R \in L^p$ for $1 < p \leq 2$ and $f \in L^1$, so $K_R * f \in L^p$ and

$$\begin{aligned} S_R f &= \mathcal{F}^{-1}\left[\chi_{(-R,R)}\hat{f}\right] \\ &= \sqrt{2\pi}\mathcal{F}^{-1}[\chi_{(-R,R)}] * \mathcal{F}^{-1}[\hat{f}] \\ &= K_R * f. \end{aligned}$$

Now suppose that $f \in L^2$ with Fourier transform $\hat{f} \in L^2$. For each $x \in \mathbb{R}$, define $\hat{g}_x \in L^1 \cap L^2$ by

$$\hat{g}_x(\xi) = \frac{1}{\sqrt{2\pi}}\chi_{(-R,R)}(\xi)e^{-ix\xi}.$$

Then $g_x \in L^2$ is given by

$$g_x(y) = K_R(x-y).$$

(Multiplication of the Fourier transform by $e^{ih\cdot\xi}$ corresponds to translation of the function by h.) By the Plancherel theorem,

$$(g_x, f)_{L^2} = (\hat{g}_x, \hat{f})_{L^2},$$

or

$$\int \overline{g_x(y)}f(y)\,dy = \int \overline{\hat{g}_x(\xi)}\hat{f}(\xi)\,d\xi,$$

which gives

$$\int K_R(x-y)f(y)\,dy = \frac{1}{\sqrt{2\pi}}\int_{-R}^{R}\hat{f}(\xi)e^{ix\xi}\,d\xi.$$

(b) If $f \in L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$ by the Plancherel theorem. Writing

$$\hat{f}_R = \chi_{(-R,R)}\hat{f}$$

where $\chi_{(-R,R)}$ is the characteristic function of $(-R, R)$, we have

$$\hat{f}_R \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

since f_R has compact support. Thus,

$$S_R f = \mathcal{F}^{-1}[\hat{f}_R]$$

is well-defined as an integral. We also have

$$\hat{f}_R \to \hat{f} \qquad \text{in } L^2(\mathbb{R}) \text{ as } R \to \infty.$$

It then follows from the Plancherel theorem that

$$S_R f \to f \qquad \text{in } L^2(\mathbb{R}) \text{ as } R \to \infty\ .$$