Satellite knots

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March 22, 2018

Abstract

Essential tori in knot complements were first studied by Horst Schubert in the 1950s. He described the satellite construction for knots and established several natural properties of this construction. This chapter considers satellite knots and how they fit into a more recent discussion of 3-manifolds, most notably JSJ decompositions and geometrization.

1 Introduction

In the 1950s Horst Schubert developed an understanding of essential tori in knot complements. His work can be interpreted as a specialized version and precursor of the work of Jaco-Shalen and Johannson in the 1970s. Specifically, Schubert described JSJ decompositions for knot complements, see [8], and proved that they were finite, see Section 6.

Let K be a knot in $\mathbb{S}^3$. An essential torus T in the complement of K is, by definition, incompressible and non peripheral in the complement of K, but necessarily compressible in $\mathbb{S}^3$. Via an exercise in Dehn's Lemma and the Schönflies theorem it follows that T bounds a solid torus containing K. Denote this solid torus by V. The complement of V is a knot complement (usually not homeomorphic to the complement of K). This consideration gives rise to the following definition:

Definition 1. *Let J be a nontrivial oriented knot in $\mathbb{S}^3$ and V a closed regular neighborhood of J. Let $\tilde{V}$ be an oriented unknotted closed solid torus in $\mathbb{S}^3$ and $\tilde{K}$ an oriented knot in the interior of $\tilde{V}$. A meridional disk of $\tilde{V}$ will meet $\tilde{K}$ in a finite subset. The least number of times a meridional disk of $\tilde{V}$ must meet $\tilde{K}$ is called the* wrapping number *of the pattern. Suppose that the wrapping number of the pattern is greater than zero and let $h : (\tilde{V}, \tilde{K}) \to (V, K)$ be an oriented homeomorphism of pairs. The image of $\tilde{K}$ under h, denoted by K, is a knot in $V \subset \mathbb{S}^3$ called a* satellite knot*.*

The knot J is called a companion knot *of K and the torus $T = \partial V$ is called a* companion torus*. The pair $(\tilde{K}, \tilde{V})$ is called a* pattern *of K.*

Note that the homeomorphism h in the above definition is unique up to isotopy. The use of the word "satellite" alludes to the fact that a satellite knot "orbits" around its companion knot.

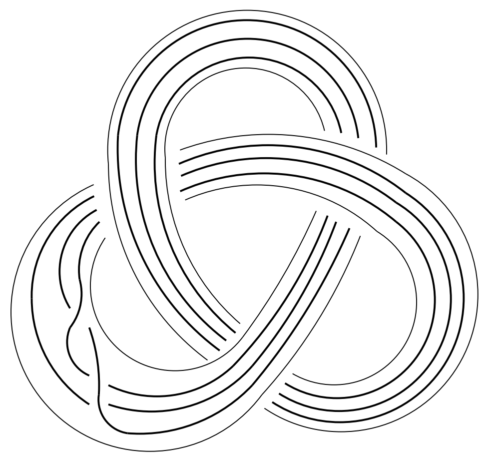

Figure 1: *A satellite knot*

2 Case studies

Three special cases deserve to the pointed out:

Definition 2. *A satellite knot with a torus knot as its pattern is called a* cabled knot*. More specifically, a satellite knot with pattern the* (p, q)*-torus knot and companion* J *is called a* (p, q)-cable of J*.*

See Figure 2.

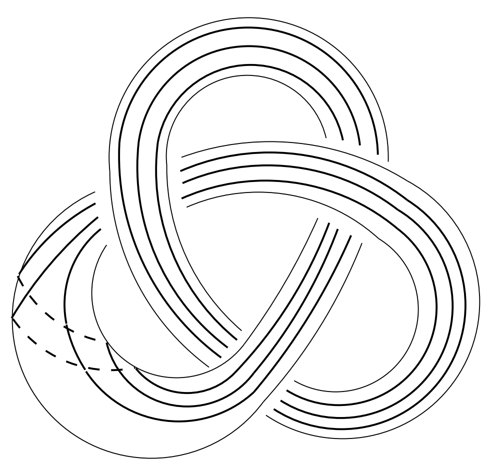

Figure 2: *The* $(3, 2)$*-cable of the trefoil*

Definition 3. *A satellite knot with a pattern of wrapping number* 1 *is called a* connected sum *of knots. The companion torus of a satellite knot with wrapping number* 1 *is called a* swallow-follow torus*.*

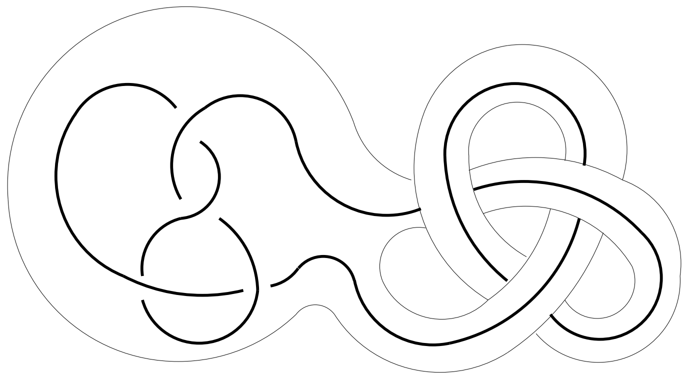

Figure 3: *A swallow-follow torus*

We must reconcile the definition just given with the standard definition, which defines the connected sum of knots $K_1 \subset \mathbb{S}^3$ and $K_2 \subset \mathbb{S}^3$ in terms of the pairwise connected sum $(K, \mathbb{S}^3) = (K_1, \mathbb{S}^3)\#(K_2, \mathbb{S}^3)$. This is accomplished by setting $K_1 = J$ and $K_2 = \tilde{K}$ or vice versa. See Figures 3 and 4 and Section 3 below.

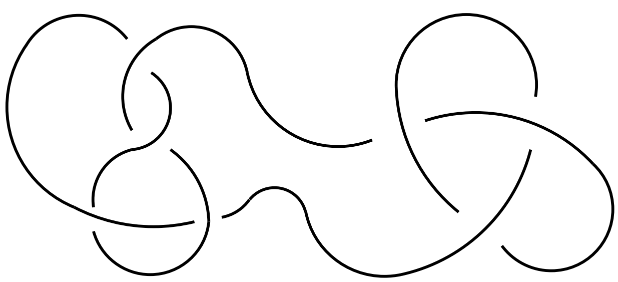

Figure 4: *The connected sum of the figure 8 knot and the trefoil*

Definition 4. *A satellite knot with the pattern pictured in Figure 5 is called a* doubled knot*.*

3 A theorem of Schubert on companion tori

In the 1950s Schubert investigated companion tori and how they lie with respect to each other.

Definition 5. *A* decomposing sphere *for a knot $K \subset \mathbb{S}^3$ is a* 2*-sphere S that meets K in exactly two points and separates K into two knotted arcs. (*I.e.*, for B the closure of a* 3*-ball complementary to S, $K \cap B$ must not be parallel into ∂B.)*

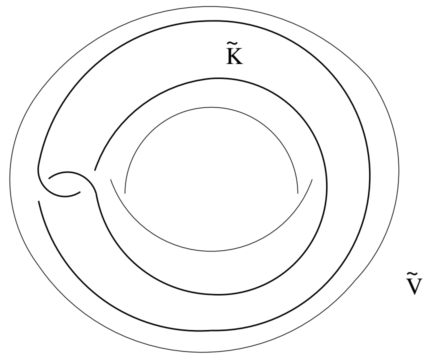

Figure 5: *The pattern for a doubled knot*

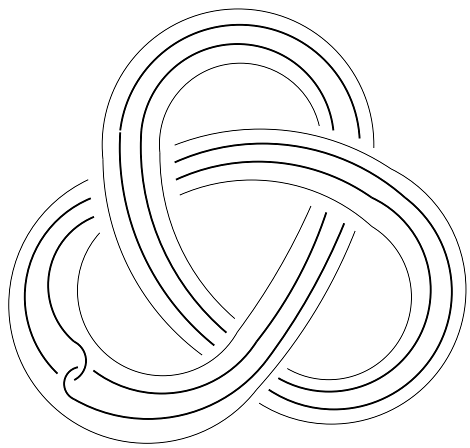

Figure 6: *The double of the trefoil*

Consider a decomposing sphere for the knot K. It is the 2-sphere along which a connected sum of knots

$$(\mathbb{S}^3, K) = (\mathbb{S}^3, K_1)\#(\mathbb{S}^3, K_2)$$

is performed. Given K, a closed regular neighborhood $N(K)$, and the decomposing sphere S, the boundary of the closure of a component of $\mathbb{S}^3 - (N(K) \cup S)$ is a torus. In fact, it is a swallow-follow torus after an isotopy. See Figures 3 and 8.

For $K = K_1\#K_2$, denote the swallow-follow torus that swallows K_1 and follows K_2 by T_1 and the swallow-follow torus that swallows K_2 and follows K_1 by T_2. We call T_1 and T_2 *complementary* swallow-follow tori. For $i = 1, 2$, T_i bounds a solid torus containing K which we denote by V_i. If both swallow-follow tori are as in Figure 8, then $T_1 \cap T_2 = \emptyset$. Interestingly, V_1 then contains the closure of the complement of V_2 and vice versa.

Unless K_1 or K_2 is the trivial knot, the swallow-follow torus T_1 will not be isotopic to T_2. If T_1 and T_2 are as in Figure 8, then they will be disjoint, but we are interested in positioning them as in Figure 3. They will then meet in two simple closed curves that are meridians in both T_1 and T_2. Assume that this is the case, then the solid tori $V_1 \cup V_2$ form an unknotted solid torus.

We build a knot satellite knot L with companion any nontrivial knot and pattern

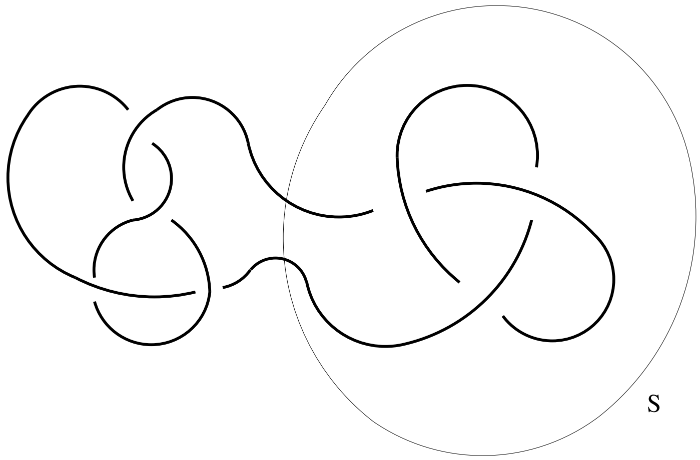

Figure 7: *The connected sum of the figure 8 knot and the trefoil with a decomposing sphere*

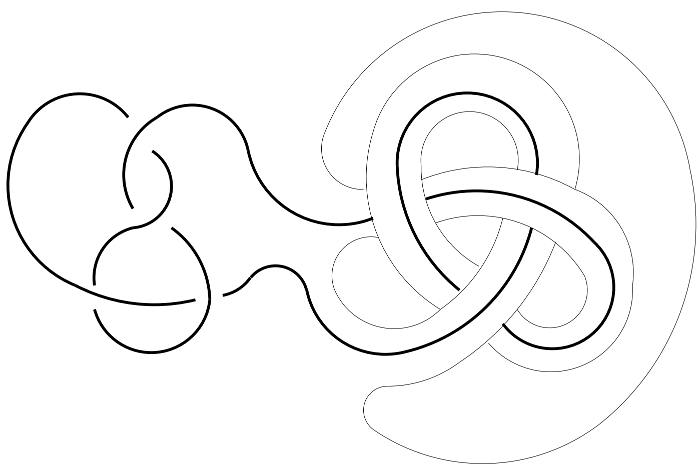

Figure 8: *A swallow-follow torus after an isotopy*

(K, *unknotted solid torus*). The images of T_1 and T_2 can no longer be isotoped to be disjoint and isotoping them to meet in two simple closed curves that are meridians in both the image of T_1 and the image of T_2 provides the best positioning of the two tori.

These best possible positionings for tori are described in the theorem below, where Schubert summarizes his findings concerning essential tori in knot complements:

Theorem 1. *(Schubert) Let T_1, T_2 be distinct companion tori of a knot K and V_1, V_2 the solid tori they bound. (*I.e.*, V_1, V_2 each contain K, $T_1 = \partial V_1$ and $T_2 = \partial V_2$.) The solid tori V_1 and V_2 can be isotoped so that (at least) one of the following holds:*

- *V_1 lies in the interior of V_2;*
- *V_2 lies in the interior of V_1;*

- *V_1 contains the closure of the complement of V_2 and V_2 contains the closure of the complement of V_1;*
- *T_1 meets T_2 in two simple closed curves that are meridians in both solid tori.*

For more information on the modern point of view on companion tori in knot complements, that is, as a characteristic submanifold of a JSJ decomposition, see [1].

4 The Whitehead manifold

The double of a knot, discussed in Section 2, is also called the Whitehead double. Indeed, note that the complement of $\tilde{K}$ in $\tilde{V}$, for $(\tilde{K}, \tilde{V})$ as pictured in Figure 5, is exactly the complement of the Whitehead link, see Figure 9.

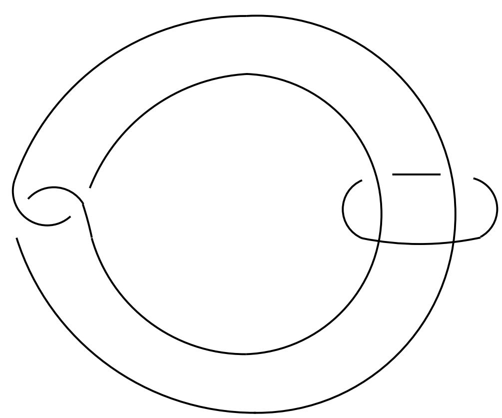

Figure 9: *The Whitehead link*

Definition 6. *The* Whitehead manifold *is the 3-manifold M^W obtained by iterating the satellite construction as follows: To begin, we choose $\tilde{K}$ from Figure 5 as our companion and also choose $(\tilde{K}, \tilde{V})$ from Figure 5 as our pattern to obtain the satellite knot K_1. In Step 2 we choose K_1 as our companion and again choose $(\tilde{K}, \tilde{V})$ from Figure 5 as our pattern to obtain the satellite knot K_2. We continue in this fashion to obtain a sequence of satellite knots $K_1, K_2, K_3, \ldots$ each of which is an unknot in $\mathbb{S}^3$. For $i = 1, 2, 3, \ldots,$ choose a closed regular neighborhood N_i of K_i such that $N_1 \supset N_2 \supset N_3 \supset \ldots$. We set*

$$N_\infty = \cap_{i=1}^{\infty} N_i$$

and

$$M^W = \mathbb{S}^3 - N_\infty.$$

We can think of the Whitehead manifold as the complement of an iterated satellite knot. Note that for $i = 1, 2, 3, \ldots$ the homomorphism

$$\pi_1(N_{i+1}) \to \pi_1(N_i)$$

is trivial. The Whitehead manifold is interesting for several reasons. Most importantly, it is contractible, yet not homeomorphic to $\mathbb{R}^3$. For more information, see [11] or [5].

5 The genus of a satellite knot

It is interesting to consider how invariants of knots behave with respect to the satellite construction.

Definition 7. *A* Seifert surface *of a knot K is a compact orientable surface $S \subset \mathbb{S}^3$ such that $\partial S = K$. The* genus *of K is the least possible genus of a Seifert surface of K.*

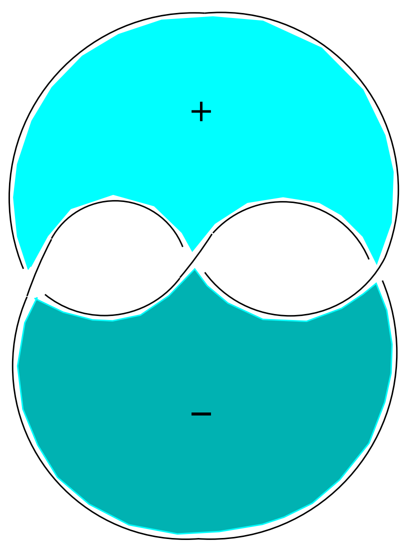

Figure 10: *A Seifert surface for the trefoil*

Figures 10 and 11 illustrate that the genus of a satellite knot is not necessarily greater than that of its companion. Nevertheless, as we shall see below, the behavior of genus vis-à-vis the satellite construction is completely understood.

Let V be a knotted solid torus in $\mathbb{S}^3$. Then V is homeomorphic to $\mathbb{D}^2 \times \mathbb{S}^1$. Under this homeomorphism, $\mathbf{0} \times \mathbb{S}^1$ is homeomorphic to a curve c called the *core* of V. Since c is a knot in $\mathbb{S}^3$, it has a Seifert surface S. After isotopy, if necessary, the Seifert surface S meets ∂V in a single simple closed curve c'. We leave it as an exercise to show that the isotopy class of c' does not depend on our choice of Seifert surface S.

Definition 8. *A simple closed curve in ∂V parallel to c' is called a* preferred longitude.

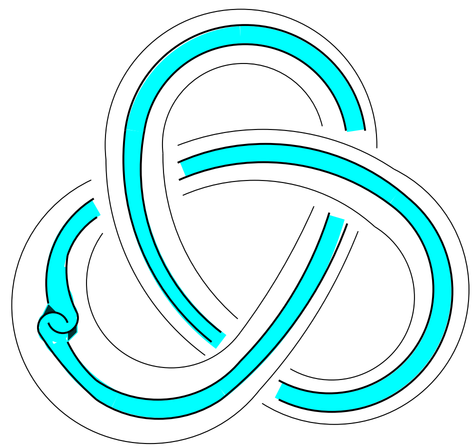

Figure 11: *A Seifert surface for the double of the trefoil*

We will build a Seifert surface for a satellite knot from a surface inside the companion torus along with Seifert surfaces for the companion torus. The surface inside the companion torus should mimic the features of a Seifert surface:

Definition 9. *Let $(\tilde{K}, \tilde{V})$ be a pattern. Orient $\tilde{K}$, $\tilde{V}$ and a meridian disk of $\tilde{V}$. Then the oriented intersection number r between $\tilde{K}$ and the meridian disk of $\tilde{V}$ is called the* winding number *of $(\tilde{K}, \tilde{V})$.*

Let $(\tilde{K}, \tilde{V})$ be a pattern with winding number r. A relative Seifert surface *is a compact orientable surface in $\tilde{V}$ whose interior is disjoint from $\tilde{K}$ and whose boundary consists of $\tilde{K}$ together with r disjoint coherently oriented preferred longitudes.*

The genus *of $(\tilde{K}, \tilde{V})$, denoted by $genus(\tilde{K}, \tilde{V})$, is the smallest possible genus of a relative Seifert surface for $(\tilde{K}, \tilde{V})$.*

It is important to distinguish the wrapping number and winding number of a pattern. For instance, the wrapping number of the pattern in Figure 5 is two, whereas the winding number is zero.

We obtain a Seifert surface for K from a relative Seifert surface for $(\tilde{K}, \tilde{V})$ by capping off the r preferred longitudes in ∂V with Seifert surfaces for the companion. We thereby obtain the following inequality:

$$genus(K) \leq r \cdot genus(J) + genus(\tilde{K}, \tilde{V})$$

Using a delicate argument, Schubert proved that the reverse inequality also holds, thereby establishing the following:

Theorem 2. *(Schubert) Let K be a satellite knot with companion J and pattern $(\tilde{K}, \tilde{V})$, then*

$$genus(K) = r \cdot genus(J) + genus(\tilde{K}, \tilde{V}).$$

6 Bridge numbers

The bridge number of a knot, denoted $b(K)$, discussed elsewhere in this volume, assigns a natural number to each knot and behaves well with respect to the satellite construction. Specifically, Schubert proves the following (see [6] or [9]):

Theorem 3. *(Schubert) Let K be a satellite knot with companion J and pattern $(\tilde{K}, \tilde{V})$ of wrapping number k. Then*

$$b(K) \geq k \cdot b(J).$$

If $K = K_1 \# K_2$, then

$$b(K) = b(K_1) + b(K_2) - 1.$$

Recall that in Theorem 1, companion tori are nested unless they are both swallow-follow tori. For a prime knot K, *i.e.*, for a knot that is not a connected sum, there are no swallow-follow tori and hence Theorem 1 tells us that companion tori are nested. Furthermore, if the prime knot K is a satellite knot (with nontrivial companion and pattern), then the wrapping number of the pattern will be at least 2. In particular, Theorem 3 then tells us that each companion knot will have bridge number strictly lower than that of K. It therefore follows that a prime knot can have only finitely many non isotopic companion tori. In this manner Schubert established finiteness, in the case of knot complements, for what later became known as JSJ decompositions.

It deserves to be pointed out that decompositions of knots into prime knots, unlike JSJ decompositions, are not canonical. They must always be finite, see [10], but collections of decomposing spheres and swallow-follow tori need not be isotopic when there are more than two prime factors. However, prime factors into which certain maximal collections of decomposing spheres decompose a knot are unique up to homeomorphism. See [7].

Ryan Budney gives a comprehensive survey of JSJ decompositions of knot and link complements, see [1]. His description is formulated in the language of graphs, using what he calls companionship graphs. In particular, he identifies precisely which graphs are companionship graphs of knots and links and what types of knot and link complements occur as basic building blocks. He also describes operations on knots and links such as cabling, connect-sum, Whitehead doubling and the deletion of a component and how these operations tie into a description of the JSJ decompositions of knot and link complements.

7 Geometrization

Thurston's Geometrization conjecture, proved by Grisha Perelman, see [2], [3], [4] tells us that every orientable 3-manifold can be decomposed along a finite collection of 2-spheres and tori into pieces each of which is geometric, *i.e.*, admits a complete finite volume Riemannian metric and has universal cover isometric to one of the following:

- $\mathbb{H}^3$
- $\mathbb{E}^3$
- $\mathbb{S}^3$
- $\mathbb{H}^2 \times \mathbb{R}$
- $\mathbb{S}^2 \times \mathbb{R}$
- The universal cover of $SL(2, \mathbb{R})$
- Nil
- $Solv$

Long before the Geometrization conjecture was proved, Thurston established its specialization to knot and link complements. Specifically, he proved that for a prime knot K, one of the following holds:

- K is a torus knot;
- K is a satellite knot;
- The complement of K admits a complete finite volume hyperbolic structure.

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