ON DISTINCT CONSECUTIVE r-DIFFERENCES.

JUNXIAN LI AND GEORGE SHAKAN

Abstract. Let $A \subset \mathbb{R}$ be finite and $D_r(A)$ be the number of distinct consecutive r-differences of A. We show $|A+B| \gg_r D_r(A)|B|^{1/(r+1)}$ for any finite $B \subset \mathbb{R}$. Utilizing de Bruijn sequences, we construct sets for which the above inequality is sharp. For the set $\{n\alpha \pmod 1\}_{1 \leq n \leq N}$, we improve immensely upon the above inequality and obtain sharp bounds for the number of distinct consecutive r-differences, generalizing Steinhaus' three gap theorem. We also consider a dual problem concerning the number of distinct consecutive r-differences of $\{T : \{T\theta\} < \phi\}$, where $\theta \in \mathbb{R}$ and $\phi \in [0,1]$, generalizing a result of Slater.

Contents

1. Introduction

Given $A, B \subset \mathbb{R}$ finite, we define the *sumset*

$$A + B = \{a + b : a \in A, b \in B\}.$$

Let $A = \{a_1 < \ldots < a_k\}$. We say A is *convex* if for all $1 < i < k$

$$a_i - a_{i-1} < a_{i+1} - a_i.$$

Hegyvári [4], answering a question of Erdös, proved that if A is convex then

$$|A + A| \gg |A| \log |A| / \log\log |A|.$$

Konyagin [5] and Garaev [3] showed if A is a convex set then

$$|A \pm A| \gg |A|^{3/2},$$

while Schoen and Shkredov improved this to

$$|A - A| \gg |A|^{8/5} \log^{-2/5} |A|, \qquad |A + A| \gg |A|^{14/9} \log^{-2/3} |A|.$$

Elekes, Nathanson, and Ruzsa [2] then showed that for any convex set A and any B,

$$|A + B| \gg |A||B|^{1/2}.$$

2010 *Mathematics Subject Classification.* Primary: 11B13, 05B10, Secondary: 11B30.
Key words and phrases. additive combinatorics, sumset estimates, convex sets .

Finally Solymosi [8], generalized this to show that if the differences $a_{i+1}-a_i$ are distinct for $1 \le i \le k-1$, then

$$|A+B| \gg |A||B|^{1/2},$$

and a construction in the same paper, due to Ruzsa, shows this bound is sharp.

Our first goal is to generalize this result of Solymosi [8]. Fix $r \ge 1$ an integer. We say a set A has *distinct consecutive r-differences* if for $1 \le i \le k-r$,

$$(a_{i+1}-a_i, \dots, a_{i+r}-a_{i+r-1})$$

are distinct.

Theorem 1.1 *Let A and B be finite subsets of real numbers with $|A|=k$ and $|B|=\ell$ and suppose A has distinct consecutive r-differences. Then*

$$|A+B| \gg e^{-r(\log 2+1)} k\ell^{1/(r+1)}.$$

The implied constant is absolute.

The case $r=1$ is Theorem 1.1 in [8]. Theorem 1.1 applies to more general sets than addressed in [8] but our bound is smaller by a power of ℓ when $r>1$. We also show below that Theorem 1.1 is best possible, up to the constant, utilizing ideas from the construction of de Bruijn sequences.

Here we study only the non-symmetric version of finding lower bounds for $|A+B|$ where A has distinct consecutive r-differences. We expect improvements to Theorem 1.1 in the case $B=A$.

Question 1.2 *What is the largest θ_r such that for every $A \subset \mathbb{Z}$ with distinct consecutive r-differences, one has*

$$|A+A| \gg_r |A|^{1+\theta_r/(r+1)}.$$

Theorem 1.1, with $B=A$, asserts that $\theta_r \ge 1$, while we provide a construction below that shows $\theta_r \le 2$. We remind the reader that any convex set has distinct consecutive 1-differences. So Question 1.2 generalizes the aforementioned question of Erdös regarding convex sets.

There is a generalization of Theorem 3 in [8] for distinct consecutive r-differences, which requires the following definition. Let $A_1, \dots, A_d$ be nonempty finite subsets of real numbers all of cardinality k. We say that $A_1, \dots, A_d$ have *distinct d-tuples of consecutive r-differences* if there exists permutations $\sigma_1, \dots, \sigma_d \in S_k$ such that the (dr)-tuples,

$$(a_{\sigma_1(i+1)}-a_{\sigma_1(i)}, \dots, a_{\sigma_1(i+r)}-a_{\sigma_1(i+r-1)}, \dots, a_{\sigma_d(i+1)}-a_{\sigma_d(i)}, \dots, a_{\sigma_d(i+r)}-a_{\sigma_d(i+r-1)})$$

are distinct for $1 \le i \le k-r$.

Theorem 1.3 *Suppose $A_1, \dots, A_d$ have distinct d-tuples of consecutive r-differences. Let $B_1, \dots B_d$ be nonempty finite sets of real numbers of cardinality $\ell_1, \dots, \ell_d$. Then*

$$|A_1+B_1| \cdots |A_d+B_d| \gg_{\beta,d} (k^{dr+1}\ell_1 \cdots \ell_d)^{1/(d(r+1))}.$$

The proof of Theorem (1.1) can be used to obtain an upper bound for the size of distinct r-differences of the set A. This upper bound is not sharp when the set A

has some additive structure. In particular, let α be a real irrational number and we consider the set of points

$$S_\alpha(N) := \{\{n\alpha\} : 1 \le n \le N\} = \{a_1 < \ldots < a_N\} \subset \mathbb{R}/\mathbb{Z}.$$

Here we identify $\mathbb{R}/\mathbb{Z}$ with $[0,1)$ and then use the natural ordering on $[0,1)$. Since $|A+A| \ll |A|$, the above theory suggests that A has few distinct consecutive r-differences. In fact, in 1957 Steinhaus conjectured that there are at most 3 distinct consecutive 1-differences in $S_\alpha(N)$. This was proved by Vera Sós in [9, 10] as well as Świerczkowski in [11]. Now we consider the set of distinct consecutive r-differences in $S_\alpha(N)$ defined via

$$D_r(S_\alpha(N)) := \{(a_{i+1}-a_i, \ldots, a_{i+r}-a_{i+r-1}) : a_i \in S_\alpha(N)\},$$

where $a_{i+N} = a_i$. Since there are at most 3 distinct 1-differences in $S_\alpha(N)$, there are at most 3^r distinct consecutive $r-$differences in $S_\alpha(N)$. However, we prove that the size of $D_r(S_\alpha(N))$ is much smaller than 3^r due to the structure of $S_\alpha(N)$.

Theorem 1.4 *There are at most $2r+1$ distinct consecutive r-differences in $S_\alpha(N)$.*

We also consider a dual problem studied by Slater in [7]. Given $\phi, \theta \in (0,1)$, let the set of returning times be

$$R_\theta(\phi) := \{T : \{T\theta\} < \phi\} = \{T_1 < T_2 < \ldots\}.$$

In [7, 6], Slater proved there are at most 3 distinct consecutive 1-differences in $R_\theta(\phi)$. We generalize this result to consecutive r-differences.

Theorem 1.5 *There are at most $2r+1$ distinct consecutive r-differences in $R_\theta(\phi)$.*

2. Distinct consecutive r-differences

In this section, we prove Theorem 1.1 as a corollary in a more general setting. Given any set A of size k, we let

$$D_r(A) = \{(a_{i+1}-a_i, \ldots, a_{i+r}-a_{i+r-1}) : 1 \le i \le k-r\}.$$

Proposition 2.1 *Let B be any set of size ℓ and A as above. Then*

$$|A+B| \gg e^{-r(\log 2+1)} D_r(A)|B|^{1/(r+1)}.$$

We remark that Theorem 1.1 follows immediately from Proposition 2.1 by observing that if A has the property of distinct consecutive r-differences, then $|D_r(A)| = k-r$.

Proof. If $|D_r(A)| \le 2r$, Proposition 2.1 is trivial, so we suppose we are not in this case.

For each $d \in D_r(A)$, we choose an $1 \le i(d) \le k-r$ so that

$$d = (a_{i(d)+1} - a_{i(d)}, \ldots, a_{i(d)+r} - a_{i(d)+r-1}).$$

Let $C = A+B$ and partition

$$C = C_1 \cup \ldots \cup C_t,$$

such that for $u < v$ every element of C_u is less than every element of C_v. The proof relies on double counting the following set

$$X = \{(i,b) : \text{There is a } 1 \le u \le t \text{ such that } a_i + b, \ldots, a_{i+r} + b \in C_u\}.$$

(Lower bound) Fix $b \in B$. Our assumption $|D_r(A)| > 2r$ will imply that $|C_u| \geq r$ for all $1 \leq u \leq t$, as will be seen by our choices for these sets later. Thus, for a fixed $1 \leq u \leq t-1$, there are at most r of the $d \in D_r(A)$ such that $a_{i(d)} + b, \ldots, a_{i(d)+r} + b$ do not all lie in the same C_u. Thus at least $D(A) - (t-1)r$ of the $d \in D_r(A)$ have the property that $a_{i(d)} + b, \ldots, a_{i(d)+r} + b$ all lie in one C_u. For each such d, we have

$$(i(d), b) \in X,$$

so that

$$(D_r(A) - (t-1)r)\ell \leq |X|.$$

(Upper bound) For each $1 \leq u \leq t$, we have that C_u contains at most $\binom{|C_u|}{r+1}$ subsets of size $r+1$. Thus

$$|X| \leq \sum_{u=1}^{t} \binom{|C_u|}{r+1}.$$

Putting these bounds together, we have

$$(D_r(A) - (t-1)r)\ell \leq \sum_{u=1}^{t} \binom{|C_u|}{r+1}.$$

We choose $t = \lfloor D_r(A)/(2r) \rfloor$ (which by assumption is at least 1) and $C_1, \ldots, C_t$ to differ in size by at most 1, which implies $||C_u| - |C|/t| \leq 1$. Proposition 2.1 follows from Stirling's formula and a straightforward calculation. □

We now give an informal sketch of a proof of Theorem 1.3 below, which is similar to Theorem 1.1. We also refer the reader to the proof of Theorem 3 in [8].

Sketch of proof of Theorem 1.3. The case $k < 2d$ is trivial, so we assume $k \geq 2d$. For $1 \leq m \leq d$, let $A_m = \{a_{m1}, \ldots, a_{mk}\}$, $B_m = \{b_{m1}, \ldots, b_{m\ell_m}\}$ and $C_m = A_m + B_m$. Partition $C_m = C_{m1} \cup \ldots \cup C_{mt_m}$ as in Proposition 2.1. Double count the number of

$$(a_{1\sigma_1(i)} + b_{1j}, \ldots, a_{1,\sigma_1(i+r)} + b_{1j}, \ldots, a_{d\sigma_d(i)} + b_{dj}, \ldots, a_{d,\sigma_d(i+r)} + b_{dj}),$$

such that $a_{m\sigma_1(i)} + b_{mj}, \ldots, a_{m,\sigma_1(i+r)} + b_{mj}$ all lie in a single C_{mu}. Similar to Theorem 3 in [8], this implies an inequality of the form

$$(k - r\sum_{m=1}^{d} t_m) \leq \sum_{u_1=1}^{|C_1|} \cdots \sum_{u_d=1}^{|C_d|} \binom{|C_{1,u_1}|}{r+1} \cdots \binom{|C_{d,u_d}|}{r+1}.$$

Choosing $t_m = \lfloor k/(2d) \rfloor$ and the C_{mj} to differ in size by at most 1 implies Theorem 1.3. □

We now show that Theorem 1.1 is best possible up to the constant. To do this we utilize a lemma from graph theory to generalize a construction due to Ruzsa as presented in [8].

Lemma 2.2 *Let S be any set. There exists a sequence $s_1, \ldots, s_k$ of elements of S (with repeats) such that*

(a) The ordered $(r+1)$-tuples $(s_j, \ldots, s_{j+r})$ are distinct for $1 \leq j \leq k$, where $s_{j+k} = s_j$,

(b) $k = |S|(|S|-1)^r$,
(c) for $1 \leq j \leq k$, $s_j \neq s_{j+1}$.

We remark that if the last condition were eliminated and k were replaced by $|S|^{r+1}$, then we would be in search of a de Bruijn sequence. These are known to exists and are well-studied. Indeed we modify a construction of de Bruijn sequences in the proof below.

Proof. We define a directed graph (V, E). We define V to be all of the $|S|(|S|-1)^{r-1}$ ordered tuples of size r with elements from S such that no two consecutive elements are the same. To define E, we say $x \to y$ if the last $r-1$ elements of x are the same (and in the same order) as the first $r-1$ elements of y. Then the outdegree and indegree of any vertex is $|S|-1$, and it is easy to see that (V, E) is strongly connected. By a standard result in graph theory, there exists an Eulerian circuit in (V, E), say $v_1, \ldots, v_k$. Setting s_j to be the first coordinate of v_j for $1 \leq j \leq k$ gives the claim. □

Now let S be any finite integer Sidon set and $s_1, \ldots, s_k$ be the sequence of elements of S as given by Lemma 2.2. We define sets $A, B \subset \mathbb{Z}^2$ via

$$A := \{(i, s_i) : 1 \leq i \leq k\}, \quad B := \{(i, 0) : 1 \leq i \leq k\}.$$

Since S is a Sidon set and by part (c) of Lemma 2.2,

$$((i+1, s_{i+1}) - (i, s_i), \ldots, (i+r, s_{i+r}) - (i+r-1, s_{i+r-1})),$$

uniquely determines

$$(s_i, \ldots, s_{i+r}).$$

By part (b) of Lemma 2.2, $(s_i, \ldots, s_{i+r})$ are distinct for $1 \leq i \leq k-r$. To achieve subsets of $\mathbb{Z}$ rather than $\mathbb{Z}^2$, we use the standard trick to define an injection $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ via

$$\phi(u, v) = Mu + v,$$

for an $M > 2(\max S - \min S)$ chosen sufficiently large so that $|\phi(A) + \phi(B)| = |A + B|$.

Thus $\phi(A)$ has the property of distinct consecutive r-differences. But

$$|\phi(A) + \phi(B)| = |A + B| \leq 2k|S| \ll |A||B|^{1/(r+1)}.$$

We remark the set $\phi(A)$ as defined above is an example that shows $\theta_r \leq 2$ in Question 1.2. That is, we have

$$|A + A| \ll |A|^{1+2/(r+1)}.$$

This follows from the plus version of Ruzsa's triangle inequality, which asserts

$$|A + A||B| \leq |A + B|^2 \ll |A|^{2+2/(r+1)}.$$

Alternatively, one could compute $|A + A|$ explicitly to see that $|A|^{1+2/(r+1)}$ is the right order of magnitude of $|A + A|$.

3. Distinct consecutive r-differences of $\{n\alpha\}$

Proof of Theorem 1.4. Recall from the introduction that

$$S_\alpha(N) := \{\{n\alpha\} : 1 \le n \le N\} = \{a_1 < \ldots < a_N\} \subset \mathbb{R}/\mathbb{Z},$$

and

$$D_r(S_\alpha(N)) := \{(a_{i+1} - a_i, \ldots, a_{i+r} - a_{i+r-1}) : a_i \in S_\alpha(N)\}.$$

To obtain an upper bound for $\#D_r(S_\alpha(N))$. We consider the set

$$\begin{aligned} D_r(\alpha, N) := \{(\{a_{i+1}\alpha\} - \{a_i\alpha\}, \cdots, \{a_{i+r}\alpha\} - \{a_{i+r-1}\alpha\}) :& \\ \{(a_i - 1)\alpha\}, \cdots, \{(a_{i+r} - 1)\alpha\}& \\ \text{are not consecutive elements in } S_\alpha(N)\},& \end{aligned}$$

which contains $D_r(S_\alpha(N))$. Thus to prove Theorem 1.4, it is enough to give an upper bound of $\#D_r(\alpha, N)$. The case when $\{a_i\alpha\}, \cdots, \{a_{i+r}\alpha\}$ are consecutive elements in $S_\alpha(N)$ while $\{(a_i-1)\alpha\}, \cdots, \{(a_{i+r}-1)\alpha\}$ are not consecutive elements in $S_\alpha(N)$ can only happen if

1. $a_j - 1 = 0$ for some $i \le j \le i + r$.
2. there exists a_k such that $\{a_k\alpha\}$ is between $\{(a_j - 1)\alpha\}$ and $\{(a_{j-1} - 1)\alpha\}$ for some $i + 1 \le j \le i + r$.

The first case happens if and only if $a_j = 1$ for some $i \le j \le i + r$. The second case happens if and only if $a_k = N$ for some $i + 1 \le k \le i + r$. Thus there are at most $2r + 1$ distinct consecutive r-differences in the sequence $S_\alpha(N)$. □

Next we give a description of the pattern of the consecutive r-differences in $S_\alpha(N)$.

Lemma 3.1 *Suppose $\{n_1\alpha\}, \{n_2\alpha\}, \cdots, \{n_k\alpha\}$ are consecutive elements in $S_\alpha(N)$. Then $\{(N + 1 - n_k)\alpha\}, \cdots, \{(N + 1 - n_2)\alpha\}, \{(N + 1 - n_1)\alpha\}$ are consecutive elements in $S_\alpha(N)$.*

Proof. The map $\{j\alpha\} \mapsto \{(N + 1 - j)\alpha\}$ is a permutation of $S_\alpha(N)$. Since $\{m\alpha\} = 1 - \{-m\alpha\}$ and $\{n_1\alpha\} < \{n_2\alpha\} < \cdots < \{n_k\alpha\}$, it follows that $\{(N + 1 - n_1)\alpha\} < \{(N + 1 - n_2)\alpha\} < \cdots < \{(N + 1 - n_k)\alpha\}$. There cannot be an m such that $\{m\alpha\}$ is between $\{(N+1-n_i)\alpha\} < \{(N+1-n_j)\alpha\}$, since it would follow that $\{(N+1-m)\alpha\}$ is in between $\{n_j\alpha\}$ and $\{n_i\alpha\}$, a contradiction. □

Corollary 3.2 *Suppose $L_1\alpha, \cdots, L_t\alpha, \alpha, R_1\alpha, \cdots, R_k\alpha \subset \mathbb{R}/\mathbb{Z}$ are the consecutive terms around $\{\alpha\}$ in $S_\alpha(N)$. Then $(N + 1 - R_k)\alpha, \cdots, (N + 1 - R_1)\alpha, N\alpha, ((N + 1 - l_t)\alpha), \cdots, (N + 1 - L_1)\alpha \subset \mathbb{R}/\mathbb{Z}$ are consecutive terms around $\{N\alpha\}$.*

Theorem 3.3 *Suppose α is irrational and N is large enough so that there the $2r + 1$ elements around α in $\mathbb{R}/\mathbb{Z}$ are all in $[0, 1)$*

$$L_1\alpha, \cdots, L_r\alpha, \alpha, R_1\alpha, \cdots, R_r\alpha,$$

Let

$$1 = p_0 < \cdots < p_i < p_{i+1} < \cdots < p_{2r},$$

be a reordering of the set

$$\{1, L_1, L_2, \cdots, L_r, N + 2 - R_1, \cdots, N + 2 - R_r\}.$$

Then $2r+1$ consecutive r-differences in $S_\alpha(N)$ are given by

$$d_r(\{p_i\alpha\}),\ i=0,1,\cdots,2r,$$

where $d_r(x)$ denote the consecutive r-difference starting from x in $S_\alpha(N)$ and

$$d_r(\{n\alpha\})=d_r(\{p_i\alpha\}),\ \textit{for } p_i\le n<p_{i+1}.$$

Proof. The $2r+1$ consecutive differences are determined by the sequence

$$L_1\alpha,\cdots,L_r\alpha,\alpha,R_1\alpha,\cdots,R_r\alpha.$$

For $r+1$ of them, the consecutive r-differences are given by $r+1$ consecutive numbers in the list. Thus $L_1,L_2,\cdots,1$ determines the $r+1$ consecutive r-differences in $S_\alpha(N)$, which are given by $d_r(\{L_t\alpha\})$ for $t=1,\cdots,r$ and $d_r(\{\alpha\})$. The remaining r of the consecutive r-differences in $S_\alpha(N)$ are determined by $r+1$ consecutive numbers around $N\alpha$. From Lemma 3.1, the r neighbours around $N\alpha$ in $\mathbb{R}/\mathbb{Z}$ are

$$(N+1-R_r)\alpha,\cdots,(N+1-R_1)\alpha,N\alpha,(N+1-L_r)\alpha,\cdots,(N+1-L_1)\alpha.$$

Thus each consecutive r-difference is given by $r+1$ of the consecutive numbers in

$$(N+1-R_r)\alpha,\cdots,(N+1-R_1)\alpha,(N+1-L_r)\alpha,\cdots,(N+1-L_1)\alpha,$$

which is determined by $(N+1-R_r)\alpha,\cdots,(N+1-R_1)\alpha$. In fact, they are given by $d_r(\{(N+2-R_l)\alpha\})$, where $l=1,\cdots,r$. In summary,

$$D_r(S_\alpha(N))=\{d_r(\{\alpha\}),d_r(\{L_1\alpha\}),\cdots,d_r(\{L_r\alpha\}),d_r(\{N{+}2{-}R_1\alpha\}),\cdots,d_r(\{N{+}2{-}R_r\alpha\})\}$$

gives the $2r+1$ consecutive r-differences in $S_\alpha(N)$, and

$$d_r(\{n\alpha\})=d_r(\{(n+m)\alpha\}),$$

as long as $n+m\le N$ and $n+m$ doesn't belong to

$$\{1,L_1,\cdots,L_r,N+2-R_1\cdots,N+2-R_r\}.$$

So for any $p_i\le n<p_{i+1}$, we have $n-p_i\ge 0$ thus $d_r(\{n\alpha\})=d_r(\{p_i\alpha\})$. $\square$

Example 3.4 *Take $\alpha=\log_{10}2$, $r=3$, and $N=100$. The r neighbours around α are*

$$74\alpha,84\alpha,94\alpha,\alpha,11\alpha,21\alpha,31\alpha\subset\mathbb{R}/\mathbb{Z}.$$

Applying Theorem 3.3,

$$\{1,71,74,81,84,91,94\}$$

determines the 7 distinct consecutive 3-differences for $S_{\log_{10}2}(100)$. And given any $1\le n\le 100$, $d_3(\{n\alpha\})$ can be found by determining which of the following intervals n belongs to

$$[1,70],[71,73],[74,80],[81,83],[84,90],[91,93],[94,100].$$

Theorem 3.5 *Let*

$$S_{\alpha,\lambda_1,\cdots,\lambda_k}(N_1,\cdots,N_k):=\{\{\alpha n_i+\lambda_i\}|1\le n_i\le N_i,i=1,\cdots,k\}.$$

There are at most $(2r+1)k$ distinct consecutive r-differences in $S_{\alpha,\lambda_1,\cdots,\lambda_k}(N_1,\cdots,N_k)$.

Proof. We sketch the proof which is similar to the case when $k = 1$ as in Theorem 1.4. Let $N = N_1 \cdots N_k$ and denote the set

$$S_{\alpha,\lambda_1,\cdots,\lambda_k}(N_1, \cdots, N_k) := \{a_1 < \ldots < a_N\}.$$

Then the distinct consecutive r-differences can be represented by the $(r+1)$-tuple $(a_i, a_{i+1}, \cdots, a_{i+r})$ such that

$$a_i - \alpha, \cdots, a_{i+r} - \alpha$$

are not consecutive elements in $S_{\alpha,\lambda_1,\cdots,\lambda_k}(N_1, \cdots, N_k)$. This can only happen if one of the coordinates of the tuple $(a_i, a_{i+1}, \cdots, a_{i+r+1})$ is of the form $\alpha + \lambda_j$ for some j, or there is a point of the form $N_j\alpha + \lambda_j$ between a_i and a_{i+1}. This gives at most $2r + 1$ r-tuples $(a_i, a_{i+1}, \cdots, a_{i+r})$ for each j. □

Theorem 3.6 *Let B be a finite subset of $\mathbb{R}/\mathbb{Z}$, then any subset A of B has at most*

$$C_r|B|^{1-\frac{1}{r+1}}\frac{|A+B|}{|B|} + r$$

distinct consecutive r-differences for some $C_r > 0$. One may choose $C_r = \frac{2r^{1-\frac{1}{r+1}}}{(r+1)!^{\frac{1}{r+1}}}$.

We omit the proof, as it is nearly identical to that of Proposition 2.1. We remark that Theorem 3.6 is a generalization of Theorem 1 in [1]. We now show that up to the constant, Theorem 3.6 is best possible. Let $S = \{1, \ldots, |S|\}$. By Lemma 2.2, there exists $s_1, \ldots, s_k$ such that

- The ordered r-tuples $(s_j, \ldots, s_{j+r-1})$ are distinct for $1 \le j \le k$, where $s_{j+k} = s_j$,
- $k = |S|(|S| - 1)^{r-1}$,
- for $1 \le j \le k$, $s_j \ne s_{j+1}$.

We define a set $A = \{a_1 < \ldots < a_k\}$ where

$$a_i := \sum_{j=1}^{i} s_j.$$

Then A has distinct consecutive r-differences. Note that $a_k \le |S|^{r+1}$, so we let $B = \{0, \ldots, N\}$ where $N = |S|^{r+1}$, so that $A \subset B$. Note that

$$|A| \asymp |S|^r, \qquad |B| = |S|^{r+1},$$

so that $|A| \asymp |B|^{1-1/(r+1)}$. To make these subsets of $\mathbb{R}/\mathbb{Z}$, we consider the map $\phi : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ via $\phi(x) = x\alpha$ for a sufficiently small α.

4. Distinct consecutive r-differences of returning times

We recall that for $0 < \phi, \theta < 1$, we have the set of returning times

$$R_\theta(\phi) = \{T : \{T\theta\} < \phi\} = \{T_1 < T_2 < \ldots\}.$$

Proof of Theorem 1.5. We prove this theorem by induction on r. Let $s \in R_\theta(\phi)$ and $d_r(s) \in \mathbb{Z}^r$ such that s is followed by

$$s + d_r(s)^{(1)}, s + d_r(s)^{(1)} + d_r(s)^{(2)}, \cdots, s + \sum_{l=1}^{r} d_r(s)^{(l)}$$

in $R_\theta(\phi)$, where $d_r(s)^{(l)}$ denotes the l^{th} coordinate of $d_r(s)$. When $r=1$, the problem was studied by Slater in [7]. Let a, b be the least positive integers such that

$$\alpha := \{a\theta\} < \phi, \ \beta := 1 - \{b\theta\} < \phi.$$

Then from the definition of a, b, we have $\phi > \max(\alpha, \beta)$ and $\phi \le \alpha + \beta$. There are three types of $d_1(s)$ given as below

$$\begin{cases} d_1(s) = a, & 0 \le \{s\theta\} < \phi - \alpha \\ d_1(s) = a + b, & \phi - \alpha \le \{s\theta\} < \beta \\ d_1(s) = b, & \beta \le \{s\theta\} < \phi. \end{cases} \tag{1}$$

This means there is a partition of $[0, \phi)$ into three intervals, each of which determines uniquely $d_1(s)$ depending where $\{s\theta\}$ lies in the interval $[0, \phi)$. Now suppose, by induction, there are at most $(2r-1)$ distinct consecutive $(r-1)$-differences in $R_\theta(\phi)$ which are determined by a partition of $[0, \phi)$ into $(2r-1)$ intervals. That is to say there are numbers $0 < g_i < \phi, \ i = 1, \cdots, 2r-2$, such that

$$0 = g_0 < g_1 \le \cdots \le g_{2r-2} < g_{2r-1} = \phi$$

gives a partition of $[0, \phi)$ into at most $(2r-1)$ intervals. There is an one-to-one correspondence between $[g_i, g_{i+1})$ and a consecutive $(r-1)$-difference in $R_\theta(\phi)$ (note that if there are less than $2r-1$ intervals then we allow $g_i = g_{i+1}$). Now we consider a consecutive r-difference in $R_\theta(\phi)$. Depending on whether $\{s\theta\}$ lies in $[0, \phi-\alpha), [\phi-\alpha, \beta)$ or $[\beta, \phi)$, s is either followed by $s+a, s+a+b, s+b$ in $R_\theta(\phi)$, respectively. Thus $\{(s + d_1(s))\theta\}$ is determined as below:

$$\begin{cases} d_1(s) = a, & \alpha \le \{(s+a)\theta\} < \phi, \\ d_1(s) = a + b, & \phi - \beta \le \{(s+a+b)\theta\} < \alpha, \\ d_1(s) = b, & 0 \le \{(s+b)\theta\} < \phi - \beta. \end{cases} \tag{2}$$

It follows that $\phi - \beta, \alpha, g_0, \ldots, g_{2r-1}$ gives rise to a partition of $[0, \phi)$ into at most $(2r+1)$ intervals, each of which corresponds uniquely to a consecutive r-difference, depending on which one of these intervals $\{(s + d_1(s))\theta\}$ lies. In fact, depending on which intervals of $[g_i, g_{i+1})$, $[0, \phi-\beta)$ (repectively $[\phi-\beta, \alpha)$, $[\alpha, \phi)$) intersect, the possible $r-1$ returning times following $(s, s+b)$ (respectively $(s, s+a+b), (s, s+a)$) will be uniquely determined.

To illustrate, we give the example of $d_2(s)$. For $d_2(s)$, there are three possibilities depending on α, β and ϕ.

$$0 \leq \phi - \alpha < \phi - \beta < \beta < \alpha < \phi :$$

$$\begin{cases} d_2(s) = (a, b), & \{s\theta\} \in [0, \phi - \alpha) \\ d_2(s) = (a+b, a+b), & \{s\theta\} \in [\phi - \alpha, 2\beta - \alpha) \\ d_2(s) = (a+b, b), & \{s\theta\} \in [2\beta - \alpha, \beta) \\ d_2(s) = (b, a), & \{s\theta\} \in [\beta, \phi - \alpha + \beta) \\ d_2(s) = (b, a+b), & \{s\theta\} \in [\phi - \alpha + \beta, \phi) \end{cases} \tag{3}$$

$$0 \leq \phi - \beta < \phi - \alpha < \alpha < \beta < \phi :$$

$$\begin{cases} d_2(s) = (a, a+b), & \{s\theta\} \in [0, \beta - \alpha) \\ d_2(s) = (a, b), & \{s\theta\} \in [\beta - \alpha, \phi - \alpha) \\ d_2(s) = (a+b, a), & \{s\theta\} \in [\phi - \alpha, \phi - 2\alpha + \beta) \\ d_2(s) = (a+b, a+b), & \{s\theta\} \in [\phi - 2\alpha + \beta, \beta) \\ d_2(s) = (b, a), & \{s\theta\} \in [\beta, \phi) \end{cases} \tag{4}$$

$$0 \leq \phi - \beta < \alpha < \phi - \alpha < \beta < \phi :$$

$$\begin{cases} d_2(s) = (a, a), & \{s\theta\} \in [0, \phi - 2\alpha) \\ d_2(s) = (a, a+b), & \{s\theta\} \in [\phi - 2\alpha, \beta - \alpha) \\ d_2(s) = (a, b), & \{s\theta\} \in [\beta - \alpha, \phi - \alpha) \\ d_2(s) = (a+b, a), & \{s\theta\} \in [\phi - \alpha, \beta) \\ d_2(s) = (b, a), & \{s\theta\} \in [\beta, \phi) \end{cases} \tag{5}$$

□

For rational θ there is a relation between the consecutive r-differences in $R_\theta(\phi)$ and $S_\theta(N)$, which can be found in [7]. Suppose $\theta = \frac{p}{q}$. Let $\alpha = \frac{p'}{q}$, where $pp' \equiv 1 \pmod{q}$. Then we have

$$\{1 \leq s \leq q : \{s\theta\} < \frac{N}{q}\} = q \cdot \{\{s'\alpha\} : 1 \leq s' \leq N\},$$

by mapping s to $s \equiv sp' \pmod{q}$. Thus the consecutive r-differences of the set

$$\{n \leq q | \ \{s\theta\} < \frac{N}{q}\}$$

are q times the consecutive r-differences of the set

$$\{\{s\alpha\}, 1 \leq s \leq N\}.$$

For general θ and ϕ, more complications will appear depending on representation of ϕ in terms of convergents of continued fraction expansion of θ.

Acknowledgments

The authors would like to thank Antal Balog, Oliver Roche-Newton and Alexandru Zaharescu for useful discussions. We also thank Boris Bukh for useful suggestions to an earlier draft.

References

[1] Antal Balog, Andrew Granville, and Jozsef Solymosi. Gaps between fractional parts, and additive combinatorics. *Q. J. Math.*, 68(1):1–11, 2017.
[2] György Elekes, Melvyn B. Nathanson, and Imre Z. Ruzsa. Convexity and sumsets. *J. Number Theory*, 83(2):194–201, 2000.
[3] M. Z. Garaev. On lower bounds for the L_1-norm of some exponential sums. *Mat. Zametki*, 68(6):842–850, 2000.
[4] N. Hegyvári. On consecutive sums in sequences. *Acta Math. Hungar.*, 48(1-2):193–200, 1986.
[5] SV Konyagin. An estimate of the l1-norm of an exponential sum. In *The Theory of Approximations of Functions and Operators. Abstracts of Papers of the International Conference, Dedicated to Stechkins 80th Anniversary, Ekaterinbourg*, pages 88–89, 2000.
[6] N. B. Slater. The distribution of the integers N for which $\theta N < \phi$. *Proc. Cambridge Philos. Soc.*, 46:525–534, 1950.
[7] Noel B. Slater. Gaps and steps for the sequence $n\theta$ mod 1. *Proc. Cambridge Philos. Soc.*, 63:1115–1123, 1967.
[8] József Solymosi. On distinct consecutive differences. *arXiv preprint math/0503069*, 2005.
[9] V. T. Sós. On the theory of diophantine approximations. I. *Acta Math. Acad. Sci. Hungar*, 8:461–472, 1957.
[10] V. T. Sós. On the distribution mod 1 of the sequence $n\alpha$. *Ann. Univ. Sci. Budapest Eötvös Sect. Math.*, 1:127–134, 1958.
[11] S Świerczkowski. On successive settings of an arc on the circumference of a circle. *Fundamenta Mathematicae*, 46(2):187–189, 1958.

University of Illinois Urbana-Champaign, Urbana, Illinois 61801
E-mail address: jli135@illinois.edu

University of Illinois Urbana-Champaign, Urbana, Illinois 61801
E-mail address: george.shakan@gmail.com