ESP
Kouba
Worksheet 17 Solutions

1.) If $Y = (\sin(\frac{x}{2}))^x$ then $\ln Y = x \cdot \ln(\sin(\frac{x}{2})) \rightarrow$

$\frac{1}{Y} Y' = x \cdot \frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})} \cdot \frac{1}{2} + \ln(\sin(\frac{x}{2})) \rightarrow$

$Y' = (\sin(\frac{x}{2}))^x \cdot [\frac{1}{2}x \cot(\frac{x}{2}) + \ln(\sin(\frac{x}{2}))]$ so for

$Y = (\sin(\frac{x}{2}))^x + 5^x \rightarrow$

$Y' = (\sin(\frac{x}{2}))^x \cdot [\frac{1}{2}x \cot(\frac{x}{2}) + \ln(\sin(\frac{x}{2}))] + 5^x \ln 5$ at $x = \pi$

$\rightarrow Y' = (1)^\pi \cdot [0 + \ln 1] + 5^\pi \ln 5 = 5^\pi \ln 5$

2.) $Y^3 + XY = 3Y^2 \rightarrow 3Y^2 Y' + XY' + Y = 6YY' \rightarrow$

$Y' = \frac{-Y}{(3Y^2 + X - 6Y)}$ at $(0, 3) \rightarrow Y' = \frac{-3}{9} = \frac{-1}{3}$;

$Y'' = \frac{(3Y^2+X-6Y)(-Y') - (-Y)(6YY' + 1 - 6Y')}{(3Y^2 + X - 6Y)^2}$ at $X = 0, Y = 3$

$Y'' = \frac{(9)(\frac{1}{3}) + (3)(18(\frac{-1}{3}) + 1 + 2)}{9^2} = \frac{-2}{27}$

3.) a.) $Y' = \sec^2 X + \frac{1}{1+X^2}$

b.) $Y' = \cos\sqrt{X} \cdot \frac{1}{2\sqrt{X}} - \frac{-1}{\sqrt{1-X}} \cdot \frac{1}{2\sqrt{X}}$

c.) $Y' = -\csc^2(\sin(5X)) \cdot \cos(5X) \cdot 5$

$+ \frac{1}{|\csc X|\sqrt{\csc^2 X - 1}} \cdot -\csc X \cdot \cot X$

d.) $Y' = \frac{1}{\arctan(\ln X)} \cdot \frac{1}{1+(\ln X)^2} \cdot \frac{1}{X}$

e.) $Y = \log_4 X + 3X \cdot \log_4 5$ —

$$y' = \frac{1}{x} \cdot \log_4 e + 3 \cdot \log_4 5$$

f.) $$y' = \frac{1}{x^2 + e^{-x}} \cdot \{2x - e^{-x}\} \cdot \log_3 e$$

g.) $$\ln y = (5+x)\{\ln(x+1) - \ln(3x-2)\} \rightarrow$$

$$\frac{1}{y} y' = (5+x)\left\{\frac{1}{x+1} - \frac{3}{3x-2}\right\} + \{\ln(x+1) - \ln(3x-2)\} \rightarrow$$

$$y' = \left(\frac{x+1}{3x-2}\right)^{5+x}\left[(5+x)\left\{\frac{1}{x+1} - \frac{3}{3x-2}\right\} + \ln\left(\frac{x+1}{3x-2}\right)\right]$$

h.) $$y = x^{e^x} \rightarrow \ln y = e^x \ln x \rightarrow$$

$$\frac{1}{y} y' = e^x \cdot \frac{1}{x} + e^x \ln x \rightarrow y' = x^{e^x} \cdot \left\{\frac{e^x}{x} + e^x \ln x\right\}$$

i.) $$x^2 \ln(xy) = xy^3 \ln(\tan y) \rightarrow$$

$$x^2 \cdot \frac{1}{xy} \cdot (xy' + y) + 2x \ln(xy) = xy^3 \cdot \frac{1}{\tan y} \cdot \sec^2 y \cdot y'$$

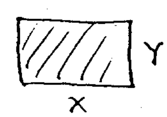
$$+ (x \cdot 3y^2 y' + y^3) \ln(\tan y) \rightarrow$$

$$\frac{x^2}{y} y' + x + 2x \ln(xy) = xy^3 \cdot \frac{\sec^2 y}{\tan y}$$

$$+ 3xy^2 \ln(\tan y) \cdot y' + y^3 \ln(\tan y) \rightarrow$$

$$y' = \frac{xy^3 \cdot \frac{\sec^2 y}{\tan y} + y^3 \ln(\tan y) - 2x \ln(xy)}{\frac{x^2}{y} - 3xy^2 \ln(\tan y)}$$

4.) a.) max. area $A = xy = x\sqrt{4-x}$



$$\rightarrow A' = x \cdot \tfrac{1}{2}(4-x)^{-1/2}(-1) + \sqrt{4-x} = 0 \rightarrow$$

$$\sqrt{4-x} = \frac{x}{2\sqrt{4-x}} \rightarrow 2(4-x) = x \rightarrow 8 = 3x \rightarrow$$

$$x = \frac{8}{3} \text{ and } y = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

A': $+$ 0 $-$ at $x = 8/3$

max. $A = 16/3\sqrt{3}$

b.) max. perimeter $P = 2x + 2y = 2x + 2\sqrt{4-x} \rightarrow$

$$P' = 2 - (4-x)^{-\frac{1}{2}} = 0 \rightarrow 2 = \frac{1}{\sqrt{4-x}} \rightarrow 4 - x = \frac{1}{4} \rightarrow$$

$$x = \frac{15}{4} \text{ and } y = \frac{1}{2}$$

P': $x=0$ $+$ 0 at $x = \frac{15}{4}$ $-$ $x=4$

max. $P = 17/2$

c.) max. $S = xy + 2x + 2y = x\sqrt{4-x} + 2x + 2\sqrt{4-x} \rightarrow$

$$S' = x \cdot \tfrac{1}{2}(4-x)^{-\frac{1}{2}} \cdot (-1) + \sqrt{4-x} + 2 - (4-x)^{-\frac{1}{2}} \rightarrow$$

$$= \frac{-x}{2\sqrt{4-x}} + \sqrt{4-x} + 2 - \frac{1}{\sqrt{4-x}} = \frac{6-3x}{2\sqrt{4-x}} + 2 = 0 \rightarrow$$

$$6 - 3x = -4\sqrt{4-x} \rightarrow 36 - 36x + 9x^2 = 16(4-x) \rightarrow$$

$$9x^2 - 20x - 28 = 0 \rightarrow$$

$$x = \frac{20 \pm \sqrt{1408}}{18} \approx 3.2$$

S': $x=0$ $+$ 0 at $x=3.2$ $-$ $x=4$

and $y = .89$

max. $S \approx 11.03$

5.) a.) $$\lim_{n \to -\infty} \left(\frac{n+2-1}{n+2}\right)^{7n} = \lim_{n \to -\infty} \left[\left(1 + \frac{1}{-(n+2)}\right)^{-(n+2)}\right]^{\frac{7n}{-(n+2)}}$$

$$= e^{-7}$$

b.) $$\lim_{n \to +\infty} \frac{1}{\left(\frac{n^3+1}{n^3}\right)^{n}} = \lim_{n \to +\infty} \frac{1}{\left[\left(1 + \frac{1}{n^3}\right)^{n^3}\right]^{\frac{1}{n^2}}} = \frac{1}{e^0} = 1$$

6.) $x''(t) = -32 \rightarrow$

$x'(t) = -32t + c$ and $x'(0) = 0 \rightarrow c = 0 \rightarrow$

$\boxed{x'(t) = -32t} \rightarrow$

$x(t) = -16t^2 + c$ and $x(0) = 5280$ ft $\rightarrow c = 5280 \rightarrow$

$\boxed{x(t) = -16t^2 + 5280}$;

a.) hit ground : $x(t) = 0 \rightarrow -16t^2 + 5280 = 0 \rightarrow$

$t = 18.17$ sec.

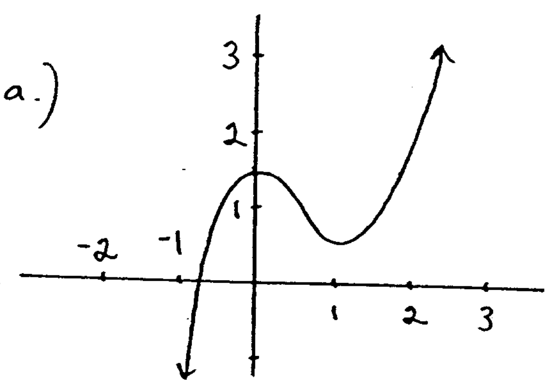
b.) distance $= (100 \text{ mph})(18.17 \text{ sec.})$

$= \left(147 \frac{\text{ft.}}{\text{sec.}}\right)(18.17 \text{ sec.})$

$= 2665$ ft.

c.) $x'(18.17) = -581.44$ ft./sec. $= -396.4$ mph.

7.) a.)



b.) $f(x) = x^3 - 2x^2 + 3/2$ is continuous with $f(0) = 3/2$ and $f(-1) = -3/2$ so by IMVT there is some # r, $-1 < r < 0$, satisfying $f(r) = 0$.

8.) $f(x) = \ln(x^4 + 1)$ is continuous on $[3,4]$ and differentiable on $(3,4)$ so by MVT there is some # c, $3 < c < 4$, satisfying

$$f'(c) = \frac{f(4) - f(3)}{4 - 3}, \text{ i.e., } \frac{4c^3}{c^4+1} = \ln 257 - \ln 82 = \ln\left(\frac{257}{82}\right)$$

9.) a.) $Y = xe^x \rightarrow$

$Y' = xe^x + e^x = e^x(x+1) = 0$

Y': $-$ 0 $+$

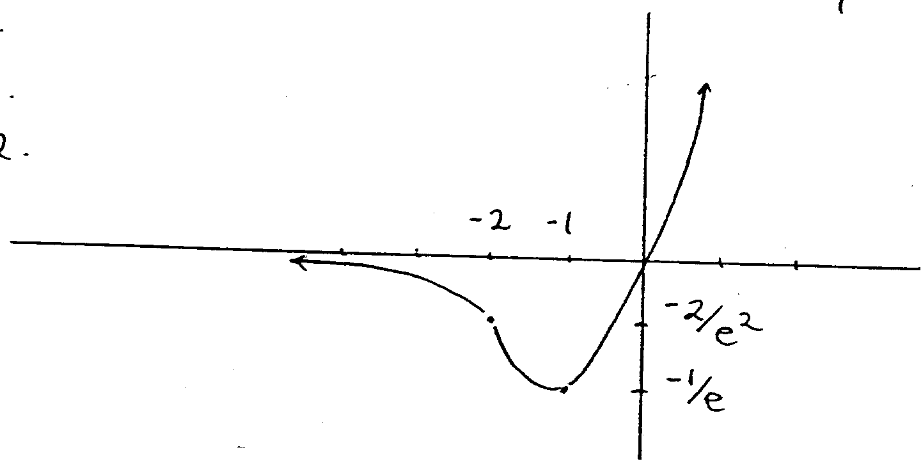
$X = -1$, $Y = -1/e$ } abs. min.

$Y'' = e^x + e^x(x+1) = e^x(x+2) = 0$

Y'': $-$ 0 $+$

$X = -2$, $Y = -2/e^2$ } inf. pt.

Y is $\uparrow$ for $X > -1$.
Y is $\downarrow$ for $X < -1$.
Y is $\cup$ for $X > -2$.
Y is $\cap$ for $X < -2$.



b.) $Y = x \ln x \rightarrow$

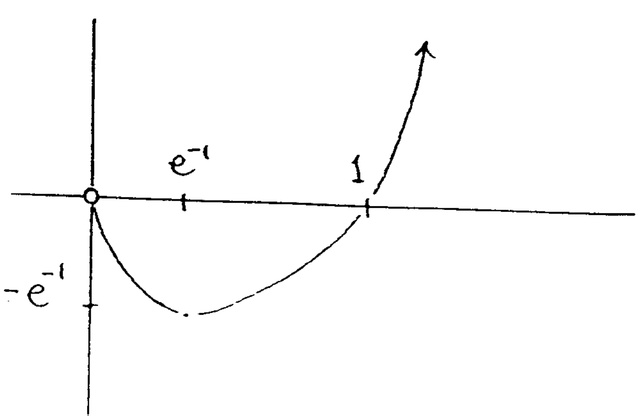
$Y' = x \cdot \frac{1}{x} + \ln x = 1 + \ln x = 0$

Y': $X=0$ $-$ 0 $+$

$X = e^{-1}$, $Y = -e^{-1}$ } abs. min.

$Y'' = \frac{1}{x}$ $\quad$ Y'': $X=0$ $+$ $+$ $+$

Y is $\uparrow$ for $X > e^{-1}$
Y is $\downarrow$ for $0 < X < e^{-1}$
Y is $\cup$ for $X > 0$



c.) $Y = e^{x} + e^{-x} \rightarrow$

$Y' = e^{x} - e^{-x} = 0$

$$\begin{array}{ccc} - & 0 & + \end{array} \quad Y'$$

$\left.\begin{array}{l} X = 0 \\ Y = 2 \end{array}\right\}$ abs. min.

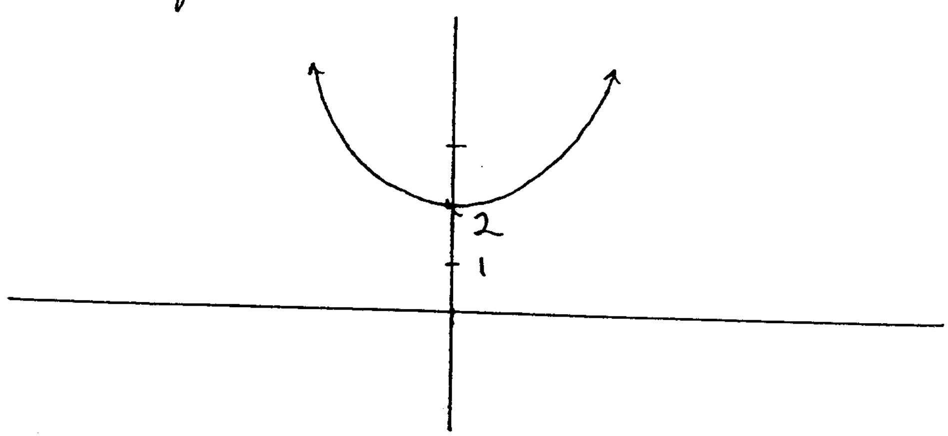
$Y'' = e^{x} + e^{-x} > 0$

$$\begin{array}{cccc} + & + & + & + \end{array} \quad Y''$$

Y is $\uparrow$ for $X > 0$.

Y is $\downarrow$ for $X < 0$.

Y is $\cup$ for all x-values.



10.) a.) $$\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$

b.) $$\lim_{x\to 1} \frac{x^2-1}{x-1} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{2x}{1} = 2$$

c.) $$\lim_{x\to 2} \frac{x^4-16}{\sqrt{x}-\sqrt{2}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 2} \frac{4x^3}{\frac{1}{2\sqrt{x}}} = \frac{32}{\frac{1}{2\sqrt{2}}} = 64\sqrt{2}$$

d.) $$\lim_{x\to 0} \frac{\tan x}{x+\sin x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{\sec^2 x}{1+\cos x} = \frac{1}{2}$$

e.) $$\lim_{x\to 1} \frac{e^{x-1}-2^{x-1}}{x^2-x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{e^{x-1}-2^{x-1}\cdot \ln 2}{2x-1} = 1-\ln 2$$

f.) $$\lim_{x\to 0} \frac{x^2\sin x + x\sin x}{x+1-\cos x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{x^2\cos x + 2x\sin x + x\cos x + \sin x}{1+\sin x}$$

$$= 0$$

g.) $$\lim_{x\to 1} \frac{x\ln x + 1 - x}{(x-1)^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{x\cdot\frac{1}{x} + \ln x - 1}{2(x-1)}$$

$$= \lim_{x\to 1} \frac{\ln x}{2(x-1)} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{2} = \frac{1}{2}$$

h.) $$\lim_{x\to \frac{\pi}{2}} \frac{\tan x}{1+\sec x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to \frac{\pi}{2}} \frac{\sec^2 x}{\sec x\cdot\tan x} = \lim_{x\to \frac{\pi}{2}} \frac{\sec x}{\tan x}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{1}{\cancel{\cos x}}\cdot\frac{\cancel{\cos x}}{\sin x} = 1$$

i.) $$\lim_{x\to +\infty} \frac{2^x+2x}{5^x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to +\infty} \frac{2^x\cdot\ln 2 + 2}{5^x\cdot\ln 5}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to +\infty} \frac{2^x(\ln 2)^2}{5^x(\ln 5)^2} = \lim_{x\to +\infty} \left(\frac{2}{5}\right)^x\cdot\frac{(\ln 2)^2}{(\ln 5)^2} = 0$$

j.) $\lim\limits_{x\to+\infty} \dfrac{x^3}{10^x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{x\to+\infty} \dfrac{3x^2}{10^x \cdot \ln 10} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{x\to+\infty} \dfrac{6x}{10^x (\ln 10)^2}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{x\to+\infty} \dfrac{6}{10^x (\ln 10)^3} = 0.$

k.) $\lim\limits_{x\to 0} \dfrac{x e^x \cos^2 6x}{e^{2x} - 1}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x\to 0} \dfrac{e^x \cos^2 6x + x e^x \cos^2 6x - x e^x \cdot 2\cos 6x \cdot \sin 6x \cdot 6}{2e^{2x}} = \dfrac{1}{2}$

l.) $\lim\limits_{x\to+\infty} \dfrac{e^x - \frac{1}{x}}{e^x + \frac{1}{x}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{x\to+\infty} \dfrac{xe^x - 1}{xe^x + 1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{x\to+\infty} \dfrac{xe^x + e^x}{xe^x + e^x}$

$= \lim\limits_{x\to+\infty} 1 = 1.$

m.) $\lim\limits_{x\to 0} \dfrac{\arcsin x}{\arctan 2x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{1+4x^2}} = \dfrac{1}{2}$

n.) $\lim\limits_{x\to 0} \left\{ \dfrac{1}{1-\cos x} - \dfrac{2}{x^2} \right\} = \lim\limits_{x\to 0} \dfrac{x^2 - 2 + 2\cos x}{x^2 (1-\cos x)}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x\to 0} \dfrac{2x - 2\sin x}{x^2 \sin x + 2x(1-\cos x)}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x\to 0} \dfrac{2 - 2\cos x}{x^2 \cos x + 2x \sin x + 2x \sin x + 2(1-\cos x)}$

$= \lim\limits_{x\to 0} \dfrac{2(1-\cos x)}{x^2 \cos x + 4x \sin x + 2(1-\cos x)}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x\to 0} \dfrac{2\sin x}{-x^2 \sin x + 2x\cos x + 4x \cos x + 4\sin x + 2\sin x}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x\to 0} \dfrac{2\cos x}{-x^2 \cos x - 2x \sin x + {}^{-}6x \sin x + 6\cos x + 6\cos x} = \dfrac{1}{6}$

o.) $\lim\limits_{x \to 0} \dfrac{\sin^2 x - x^2}{(e^{x^2}-1)^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x \to 0} \dfrac{\overbrace{2\sin x \cos x}^{\sin 2x} - 2x}{2(e^{x^2}-1)e^{x^2} \cdot 2x}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x \to 0} \dfrac{2\cos 2x \quad - 2}{4(e^{x^2}-1)e^{x^2} + 4x \cdot e^{x^2} \cdot 2x \cdot e^{x^2} + 4x(e^{x^2}-1) \cdot e^{x^2} \cdot 2x}$

$= \lim\limits_{x \to 0} \dfrac{\cancel{2}(\cos 2x - 1)}{2\cancel{4}\, e^{x^2} \cdot [4x^2 e^{x^2} + e^{x^2} - 2x^2 - 1]} = \dfrac{0}{-4} = 0$

p.) $Y = (\ln x)^{\frac{1}{x}} \to \ln Y = \frac{1}{x} \ln(\ln x) = \dfrac{\ln(\ln x)}{x}$ so

$\lim\limits_{x \to +\infty} \ln Y = \lim\limits_{x \to +\infty} \dfrac{\ln(\ln x)}{x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{x \to +\infty} \dfrac{1}{\ln x} \cdot \dfrac{1}{x} = 0$

so if $\lim\limits_{x \to +\infty} \ln Y = \ln\left(\lim\limits_{x \to +\infty} Y\right) = 0$

then $\lim\limits_{x \to +\infty} Y = 1$.

q.) $Y = (\sin x)^{1/x} \to \ln y = \frac{1}{x} \ln(\sin x) = \dfrac{\ln(\sin x)}{x}$

so $\lim\limits_{x \to 0^+} \ln Y = \lim\limits_{x \to 0^+} \dfrac{\ln(\sin x)}{x} = \dfrac{-\infty}{0^+} = -\infty$

then $\ln\left(\lim\limits_{x \to 0^+} Y\right) = -\infty$ and

$\lim\limits_{x \to 0^+} Y = 0$.

r.) $Y = (1+x)^{1/x} \to \ln Y = \dfrac{\ln(1+x)}{x} \to$

$\lim\limits_{x \to 0} \ln Y = \lim\limits_{x \to 0} \dfrac{\ln(1+x)}{x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x \to 0} \dfrac{1}{1+x} = 1$ so

$\ln\left(\lim_{x\to 0} Y\right) = 1$ and $\lim_{x\to 0} Y = e$.

s.) $Y = \left(1+\frac{5}{n}\right)^{5n} \rightarrow \ln Y = 5n \ln(1+5/n) = \dfrac{\ln(1+5/n)}{\frac{1}{5n}}$

$\lim_{n\to+\infty} \ln Y = \lim_{n\to+\infty} \dfrac{\ln(1+5/n)}{\frac{1}{5n}} \overset{"\frac{0}{0}"}{=} \lim_{n\to\infty} \dfrac{\left(\frac{1}{1+5/n}\right)\cdot\frac{-5}{n^2}}{\frac{-1}{5n^2}} = 25$

so $\ln\left(\lim_{n\to+\infty} Y\right) = 25$ and $\lim_{n\to+\infty} Y = e^{25}$.

t.) $Y = (1+n)^{1/n} \rightarrow \ln Y = \dfrac{\ln(1+n)}{n} \rightarrow$

$\lim_{n\to+\infty} \ln Y = \lim_{n\to+\infty} \dfrac{\ln(1+n)}{n} \overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to+\infty} \dfrac{1}{1+n} = 0$ so

$\ln\left(\lim_{n\to+\infty} Y\right) = 0$ and $\lim_{n\to+\infty} Y = 1$.

u.) $\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x^2}} \overset{"\frac{\infty}{\infty}"}{=} \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{\frac{-2}{x^3}}$

$= \lim_{x\to 0^+} \dfrac{-x^2}{2} = 0$.

v.) $Y = (\tan x)^{\sqrt{x/3}} \rightarrow \ln Y = \sqrt{\frac{x}{3}}\, \ln(\tan x) = \dfrac{\ln(\tan x)}{\frac{\sqrt{3}}{\sqrt{x}}} \rightarrow$

$\lim_{x\to 0^+} \ln Y = \lim_{x\to 0^+} \dfrac{\ln(\tan x)}{\frac{\sqrt{3}}{\sqrt{x}}} \overset{"\frac{\infty}{\infty}"}{=} \lim_{x\to 0^+} \dfrac{\frac{\sec^2 x}{\tan x}}{\frac{-\sqrt{3}}{2x^{3/2}}}$

$= \lim_{x\to 0^+} \left(\sqrt{x}\cdot\frac{x}{\sin x}\cdot \sec x \cdot \frac{2}{-\sqrt{3}}\right) = 0\cdot 1\cdot 1\cdot \frac{-2}{\sqrt{3}} = 0$ so

$\ln\left(\lim_{x\to 0^+} Y\right) = 0$ and $\lim_{x\to 0^+} Y = e^0 = 1$.