ESP
Kouba
Worksheet 10 Solutions

1.) a.) $\int \frac{1}{1+x^2}\,dx = \arctan x + c$

b.) $\int \frac{1}{1-x^2}\,dx = \int \left[\frac{A}{1-x} + \frac{B}{1+x}\right] dx = \int \left[\frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\right] dx$

$= -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + c$

c.) $\int \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\ln(1+e^{2x}) + c$

d.) $\int \frac{e^x}{1+(e^x)^2}\,dx = \arctan(e^x) + c$

e.) $\int \frac{1}{x(1+(\ln x)^2)}\,dx = \arctan(\ln x) + c$

f.) $\int \frac{\sec 3x \tan 3x}{1+(\sec 3x)^2}\,dx = \frac{1}{3}\arctan(\sec 3x) + c$

g.)

$$\begin{array}{r|l} & e^x \\ \hline 1+e^x & (e^x)^2 \\ & \underline{(e^x)^2 + e^x} \\ & -e^x \end{array}$$

$\int \frac{e^{2x}}{1+e^x}\,dx = \int \left[e^x - \frac{e^x}{1+e^x}\right] dx$

$= e^x - \ln(1+e^x) + c$

h.) $\int \csc(1-2x)\,dx = \frac{-1}{2}\ln|\csc(1-2x) - \cot(1-2x)| + c$

i.) $\int \frac{\sec x}{\tan x}\,dx = \int \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}\,dx = \int \csc x\,dx$

$= \ln|\csc x - \cot x| + c$

j.) $\int \frac{\sec 3x \tan 3x}{\sec^2 3x - 1}\,dx = \int \frac{\sec 3x \tan 3x}{\tan^2 3x}\,dx$

$= \int \frac{\sec 3x}{\tan 3x}\,dx = \int \csc 3x\,dx$

$= \frac{1}{3}\ln|\csc 3x - \cot 3x| + c$

k.) $\int \sin x \cos x \, dx = \int \frac{1}{2} \sin 2x \, dx = \frac{-1}{4} \cos 2x + c$

OR $\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + c$

OR $\int \sin x \cos x \, dx = \frac{-1}{2} \cos^2 x + c$

l.) $\int \cos^3 x \cdot \sin x \, dx = \frac{-1}{4} \cos^4 x + c$

m.) $\int (\sin x \cos x)^2 \, dx = \int \left(\frac{1}{2} \sin 2x\right)^2 dx$

$= \frac{1}{4} \int \sin^2 2x \, dx = \frac{1}{4} \int \frac{1 - \cos 4x}{2} \, dx$

$= \frac{1}{8}\left(x - \frac{1}{4} \sin 4x\right) + c$

n.) $\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$

$= \int [\cos x - \sin^2 x \cos x] \, dx = \sin x - \frac{1}{3} \sin^3 x + c$

o.) $\int \tan 7x \, dx = \frac{1}{7} \ln|\sec 7x| + c$

p.) $\int \tan 7x \cdot \sec^2 7x \, dx = \frac{1}{14} \tan^2 7x + c$

q.) $\int \tan^2 7x \cdot \sec^2 7x \, dx = \frac{1}{21} \tan^3 7x + c$

r.) $\int \sec^4 x \tan x \, dx = \int \sec^3 x \, (\sec x \tan x) \, dx$

$= \frac{1}{4} \sec^4 x + c$

s.) $\int \sec^5 x \tan^3 x \, dx = \int \sec^5 x \cdot \tan^2 x \cdot \tan x \, dx$

$= \int \sec^5 x \cdot (\sec^2 x - 1) \cdot \tan x \, dx$

$= \int [\sec^6 x \, (\sec x \tan x) - \sec^4 x \, (\sec x \tan x)] \, dx$

$$= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c$$

t.) $\int \tan^3 x \, dx = \int \tan x \cdot \tan^2 x \, dx$

$$= \int \tan x\,(\sec^2 x - 1)\,dx = \int (\tan x \sec^2 x - \tan x)\,dx$$

$$= \frac{1}{2}\tan^2 x - \ln|\sec x| + c$$

u.) $\int \sec^3 x \, dx = \int \sec x \cdot \sec^2 x \, dx$

$$= \int \sec x\,(1 + \tan^2 x)\,dx = \int (\sec x + \sec x \tan^2 x)\,dx$$

$$= \ln|\sec x + \tan x| + \int (\sec x \cdot \tan x)\tan x \, dx$$

(let $u = \tan x$, $dv = \sec x \tan x \, dx$, $du = \sec^2 x \, dx$, $v = \sec x$)

$$= \ln|\sec x + \tan x| + \sec x \tan x - \int \sec^3 x \, dx \Rightarrow$$

$$2\int \sec^3 x \, dx = \ln|\sec x + \tan x| + \sec x \tan x + c \Rightarrow$$

$$\int \sec^3 x \, dx = \frac{1}{2}\ln|\sec x + \tan x| + \frac{1}{2}\sec x \tan x + c.$$

v.) $3\int \frac{2x+4}{x^2+4x+7}\,dx = 3\ln|x^2+4x+7| + c$

w.) $\int \frac{1}{x^2+4x+7}\,dx = \int \frac{1}{(x+2)^2+(\sqrt{3})^2}\,dx = \frac{1}{\sqrt{3}}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + c$

x.) $\int \frac{x}{x^2+4x+7}\,dx = \int \frac{x+2-2}{x^2+4x+7}\,dx$

$$= \int \frac{x+2}{x^2+4x+7}\,dx + \int \frac{-2}{(x+2)^2+(\sqrt{3})^2}\,dx$$

$$= \frac{1}{2}\ln\left|x^2+4x+7\right| - \frac{2}{\sqrt{3}}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + c$$

Y.)

$$\begin{array}{r|l} & \quad 1 \\ x^2+4x+7 & x^2 \\ & x^2+4x+7 \\ \cline{2-2} & \quad -4x-7 \end{array}$$

so

$$\int \frac{x^2}{x^2+4x+7}\,dx = \int\left[1 - \frac{4x+7}{x^2+4x+7}\right]dx$$

$$= x - \int \frac{4x+8\;-1}{x^2+4x+7}\,dx = x - \int\left[\frac{2(2x+4)}{x^2+4x+7} - \frac{1}{(x+2)^2+(\sqrt{3})^2}\right]dx$$

$$= x - 2\ln\left|x^2+4x+7\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + c$$

Z.) $\int\left[1+\frac{4}{x}+\frac{7}{x^2}\right]dx = x + 4\ln|x| - \frac{7}{x} + c$

A.) $\int \frac{\sin\theta}{1+\cos\theta}\,d\theta = -\ln\left|1+\cos\theta\right| + c$

B.) $\int \frac{1}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}\,d\theta = \int \frac{1-\cos\theta}{1-\cos^2\theta}\,d\theta$

$$= \int \frac{1-\cos\theta}{\sin^2\theta}\,d\theta = \int\left[\csc^2 x - \csc x\cot x\right]dx$$

$$= -\cot x + \csc x + c$$

C.) $\int \frac{1}{\sec\theta+1}\cdot\frac{\sec\theta-1}{\sec\theta-1}\,d\theta = \int \frac{\sec\theta-1}{\sec^2\theta-1}\,d\theta$

$$= \int \frac{\sec\theta-1}{\tan^2\theta}\,d\theta = \int \frac{\frac{1}{\cos\theta}-1}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \int\left(\csc x\cot x - \cot^2 x\right)dx$$

$$= -\csc x - \int \cot^2 x\, dx = -\csc x - \int (\csc^2 x - 1)\, dx$$

$$= -\csc x - (-\cot x - x) + c$$

$$= -\csc x + \cot x + x + c$$

D.) $$\int \frac{\sec\theta \tan\theta}{\sec\theta + 1}\, d\theta = \ln|\sec\theta + 1| + c$$

E.) $$\int \frac{1}{\tan\theta + 1}\, d\theta = \int \frac{1}{\frac{\sin\theta}{\cos\theta} + 1}\, d\theta = \int \frac{\cos\theta}{\cos\theta + \sin\theta}\, d\theta$$

$$= \int \frac{\cos\theta}{\cos\theta + \sin\theta} \cdot \frac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta}\, d\theta = \int \frac{\cos^2\theta - \sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}\, d\theta$$

$$= \int \frac{\frac{1 + \cos 2\theta}{2} - \frac{1}{2}\sin 2\theta}{\cos 2\theta}\, d\theta = \frac{1}{2}\int \frac{1 + \cos 2\theta - \sin 2\theta}{\cos 2\theta}\, d\theta$$

$$= \frac{1}{2}\int \left[\sec 2\theta + 1 - \tan 2\theta\right] d\theta$$

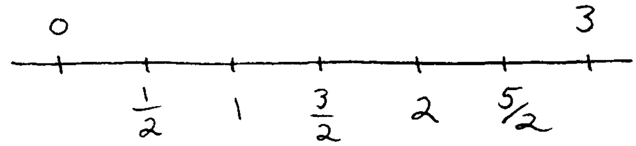
$$= \frac{1}{2}\left(\cdot \frac{1}{2}\ln|\sec 2\theta + \tan 2\theta| + \theta - \frac{1}{2}\ln|\sec 2\theta|\right) + c$$

F.) $$\int_{\frac{2\pi}{3}}^{\pi} \sqrt{\tan^2\theta}\, d\theta = \int_{\frac{2\pi}{3}}^{\pi} |\tan\theta|\, d\theta = \int_{\frac{2\pi}{3}}^{\pi} -\tan\theta\, d\theta$$

$$= -\ln|\sec\theta| \Big|_{\frac{2\pi}{3}}^{\pi} = \ln|\cos\theta| \Big|_{\frac{2\pi}{3}}^{\pi}$$

$$= \ln|\cos\pi| - \ln\left|\cos\frac{2\pi}{3}\right| = \ln 1 - \ln\frac{1}{2} = \ln 2$$

2.) a.)

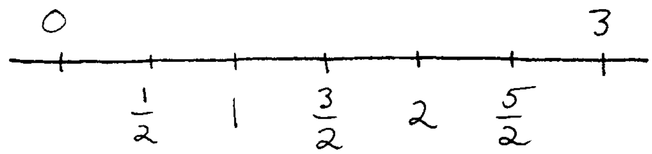


$f(x) = \dfrac{x}{1+x^4}$

$$T_6 = \frac{3-0}{2(6)}\left[f(0) + 2f\left(\tfrac{1}{2}\right) + 2f(1) + 2f\left(\tfrac{3}{2}\right) + 2f(2) + 2f\left(\tfrac{5}{2}\right) + f(3)\right]$$

$$= \frac{1}{4}\left[0 + \frac{16}{17} + 1 + \frac{48}{97} + \frac{4}{17} + \frac{80}{641} + \frac{3}{82}\right] \approx .70817$$

b.)



$f(x) = \dfrac{x}{1+x^4}$

$$S_6 = \frac{3-0}{3(6)}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + f(3)\right]$$

$$= \frac{1}{6}\left[0 + \frac{32}{17} + 1 + \frac{96}{97} + \frac{4}{17} + \frac{160}{641} + \frac{3}{82}\right] \approx .73225$$

c.) $\displaystyle\int_0^3 \frac{x}{1+(x^2)^2}\,dx = \frac{1}{2}\arctan(x^2)\Big|_0^3$

$$= \frac{1}{2}\arctan 9 - \frac{1}{2}\arctan 0 \approx .73007$$

3.) a.) $f''(x) = e^{-x^2}(4x^2 - 2)$

$f^{(4)}(x) = e^{-x^2}(16x^4 - 48x^2 + 12)$

b.) $M_2 = \displaystyle\max_{0 \le x \le 1} |f''(x)| \le e^0 \cdot |4-2| = 2$

$M_4 = \displaystyle\max_{0 \le x \le 1} |f^{(4)}(x)| \le e^0 |16 - 48 + 12| = 20$

c.) i.) $\text{Err} = \dfrac{(b-a)M_2 h^2}{12} = \dfrac{1}{6n^2} < .00001 \Rightarrow$

$n \geq 121$.

ii.) $\text{Err} = \dfrac{(b-a)M_4 h^4}{180} = \dfrac{1}{9n^4} < .00001 \Rightarrow$

$n \geq 11$.

4.)

$$2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \ddots}}}} = x \Rightarrow$$

$$2 + \frac{2}{x} = x \Rightarrow x^2 - 2x - 2 = 0 \Rightarrow$$

$$x = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 + 2\sqrt{3}}{2} = 1 + \sqrt{3} .$$