ESP Kouba

Worksheet 15 Solutions

1.) a.) $\int_1^\infty \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^\infty = \lim_{A\to\infty} \frac{-1}{x}\Big|_1^A = \lim_{A\to\infty}\left(\frac{-1}{A}-\frac{-1}{1}\right) = 1$

b.) $\int_0^1 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_0^1 = \lim_{A\to 0^+} \frac{-1}{x}\Big|_A^1 = \lim_{A\to 0^+}\left(\frac{-1}{1}-\frac{-1}{A}\right) = \infty$

c.) $\int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = \lim_{A\to\infty} \frac{-1}{e^x}\Big|_0^A = \lim_{A\to\infty}\left(\frac{-1}{e^A}-\frac{-1}{1}\right) = 1$

d.) $\int_0^\infty \frac{1}{1+x^2}\,dx = \arctan x\Big|_0^\infty = \lim_{A\to\infty} \arctan x\Big|_0^A$

$= \lim_{A\to\infty}(\arctan A - \arctan 0) = \frac{\pi}{2}$

e.) $\int_1^3 \frac{1}{x-1}\,dx = \ln|x-1|\Big|_1^3 = \lim_{A\to 1^+} \ln|x-1|\Big|_A^3$

$= \lim_{A\to 1^+}(\ln 2 - \ln|A-1|) = \infty$

f.) $\int_3^\infty \frac{1}{x+1}\,dx = \ln|x+1|\Big|_3^\infty = \lim_{A\to\infty} \ln|x+1|\Big|_3^A$

$= \lim_{A\to\infty}(\ln|A+1| - \ln 4) = \infty$

g.) $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}\Big|_0^1 = \lim_{A\to 1^-} -\sqrt{1-x^2}\Big|_0^A$

$= \lim_{A\to 1^-}\left(-\sqrt{1-A^2} - {}^-1\right) = 1$

h.) $\int_1^\infty \frac{x}{\sqrt{x^2-1}}\,dx = \int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx + \int_2^\infty \frac{x}{\sqrt{x^2-1}}\,dx = B + C$; $B = \lim_{A\to 1^+} \sqrt{x^2-1}\Big|_A^2$

$= \lim_{A\to 1^+}\left(\sqrt{3} - \sqrt{A^2-1}\right) = \sqrt{3} - 0 = \sqrt{3}$; $C = \lim_{A\to+\infty} \sqrt{x^2-1}\Big|_2^A$

$= \lim_{A\to+\infty}\left(\sqrt{A^2-1} - \sqrt{3}\right) = \infty$, so $\int_1^\infty \frac{x}{\sqrt{x^2-1}}\,dx$ diverges

i.) $\int_0^1 \ln x\,dx = (x\ln x - x)\Big|_0^1 = \lim_{A\to 0^+} (x\ln x - x)\Big|_A^1$

$$= \lim_{A\to 0^+} \left[(\ln 1 - 1) - (A\ln A - A)\right] = \lim_{A\to 0^+}\left[(A-1) - \overset{\frac{\infty}{\infty}}{\frac{\ln A}{\frac{1}{A}}}\right]$$

$$= -1 - \lim_{A\to 0^+} \frac{\frac{1}{A}}{\frac{-1}{A^2}} = -1 - \lim_{A\to 0^+}(-A) = -1$$

j.) $\displaystyle\int_1^e \frac{1}{x\ln x}\,dx = \ln|\ln x|\,\Big|_1^e = \lim_{A\to 1^+} \ln|\ln x|\,\Big|_A^e$

$$= \lim_{A\to 1^+}\left(\ln|\ln e| - \ln|\ln A|\right) = \infty$$

k.) $\displaystyle\int_0^\infty \arctan x\,dx$ $\quad$ (let $u = \arctan x$, $dv = dx$

$du = \frac{1}{1+x^2}\,dx$, $v = x$)

$$= x\arctan x\,\Big|_0^\infty - \int_0^\infty \frac{x}{1+x^2}\,dx$$

$$= x\arctan x\,\Big|_0^\infty - \frac{1}{2}\ln(x^2+1)\,\Big|_0^\infty$$

$$= \lim_{A\to\infty}\left\{x\arctan x - \frac{1}{2}\ln(x^2+1)\right\}\Big|_0^A$$

$$= \lim_{A\to\infty}\left\{A\arctan A - \frac{1}{2}\ln(A^2+1)\right\} = \infty - \infty \ ?$$

$$= \lim_{A\to\infty}\left\{\frac{\arctan A}{\frac{1}{A}} - \frac{1}{2}\ln(A^2+1)\right\}$$

$$= \lim_{A\to\infty}\left\{\frac{\arctan A - \overset{\frac{\infty}{\infty}}{\frac{\ln(A^2+1)}{2A}}}{\frac{1}{A}}\right\}$$

$$= \lim_{A\to\infty}\left\{\frac{\arctan A - \frac{A}{A^2+1}}{\frac{1}{A}}\right\} = \frac{\frac{\pi}{2} - 0}{0^+} = \infty$$

l.) $\int_{-\infty}^{\infty} \frac{1}{(x+1)^2}\,dx = \int_{-\infty}^{-1} (x+1)^{-2}dx + \int_{-1}^{\infty} (x+1)^{-2}dx$

$= \quad C \quad + \quad D$;

$C = \int_{-\infty}^{-1} (x+1)^{-2}dx = \int_{-\infty}^{-2} (x+1)^{-2}dx + \int_{-2}^{-1} (x+1)^{-2}dx$

$= \lim_{A\to-\infty} \frac{-1}{x+1}\Big|_A^{-2} + \lim_{B\to -1^-} \frac{-1}{x+1}\Big|_{-2}^{B}$

$= \lim_{A\to-\infty} \left(1 - \frac{-1}{A+1}\right) + \lim_{B\to -1^-} \left(\frac{-1}{B+1} - 1\right) = 1 + \infty$

so $\int_{-\infty}^{\infty} \frac{1}{(x+1)^2}\,dx$ diverges .

m.) $\int_{-\infty}^{\infty} \frac{1}{(x+1)^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{(x+1)^2+1}\,dx + \int_{0}^{\infty} \frac{1}{(x+1)^2+1}\,dx$

$= \lim_{A\to-\infty} \arctan(x+1)\Big|_A^0 + \lim_{B\to\infty} \arctan(x+1)\Big|_0^B$

$= \lim_{A\to-\infty} (\arctan 1 - \arctan(A+1)) + \lim_{B\to\infty} (\arctan(B+1) - \arctan 1)$

$= \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \frac{\pi}{4} = \pi$.

n.) $\int_0^3 \frac{1}{(x-1)(x-2)}\,dx = \int_0^1 \frac{1}{(x-1)(x-2)}\,dx + \int_1^2 \frac{1}{(x-1)(x-2)}\,dx$

$+ \int_2^3 \frac{1}{(x-1)(x-2)}\,dx = C + D + E$;

$C = \int_0^1 \left[\frac{-1}{x-1} + \frac{1}{x-2}\right] dx = (-\ln|x-1| + \ln|x-2|)\Big|_0^1$

$= \lim_{A\to 1^-} \{(-\ln|A-1| + \ln|A-2|) - (-\ln 1 + \ln 2)\} = \infty$

so $\int_0^3 \frac{1}{x^2-3x+2}\,dx$ diverges.

o.) $\left|\int_2^\infty \frac{\sin x}{x^2}\,dx\right| \le \int_2^\infty \frac{|\sin x|}{x^2}\,dx \le \int_2^\infty \frac{1}{x^2}\,dx$

$= \frac{-1}{x}\Big|_2^\infty = \lim_{A\to\infty} \frac{-1}{x}\Big|_2^A = \lim_{A\to\infty}\left(\frac{-1}{A} - \frac{-1}{2}\right) = \frac{1}{2}$, so

$\frac{-1}{2} \le \int_2^\infty \frac{\sin x}{x^2}\,dx \le \frac{1}{2}$.

p.) $\int_1^\infty \frac{\cos x}{x}\,dx$ (let $u = \frac{1}{x}$, $dv = \cos x\,dx$

$du = \frac{-1}{x^2}\,dx$, $v = \sin x$)

$= \frac{\sin x}{x}\Big|_1^\infty + \int_1^\infty \frac{\sin x}{x^2}\,dx$

$= \lim_{A\to\infty} \frac{\sin x}{x}\Big|_1^A + \int_1^\infty \frac{\sin x}{x^2}\,dx$

$= \lim_{A\to\infty}\left(\frac{\sin A}{A} - \sin 1\right) + \int_1^\infty \frac{\sin x}{x^2}\,dx$

$= (0 - \sin 1) + \int_1^\infty \frac{\sin x}{x^2}\,dx$;

$\left|\int_1^\infty \frac{\sin x}{x^2}\,dx\right| \le \int_1^\infty \frac{|\sin x|}{x^2}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^\infty = 1$

so that $-1 \le \int_1^\infty \frac{\sin x}{x^2}\,dx \le 1$ and

hence $\int_1^\infty \frac{\cos x}{x}\,dx$ converges.

q.) $\int_0^\infty \cos^2 3x\,dx = \int_0^\infty \frac{1+\cos 6x}{2}\,dx$

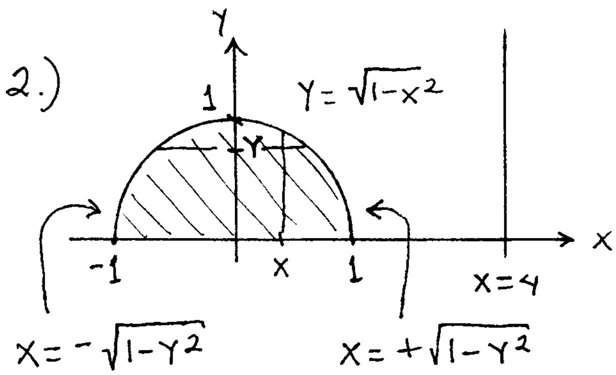
$= \frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right)\Big|_0^\infty = \lim_{A\to\infty} \frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right)\Big|_0^A$

$$= \lim_{A \to \infty} \frac{1}{2}\left(A + \frac{1}{6}\sin 6A\right) = \infty .$$

n.) $$\int_1^\infty x e^{-x}\,dx = \left(-e^{-x} - x e^{-x}\right)\Big|_1^\infty = \lim_{A\to\infty} \frac{-(1+x)}{e^x}\Big|_1^A$$

$$= \lim_{A\to\infty}\left(\frac{-(1+A)}{e^A} - \frac{-2}{e}\right) = \lim_{A\to\infty}\left(\frac{-1}{e^A} + \frac{2}{e}\right) = \frac{2}{e}$$

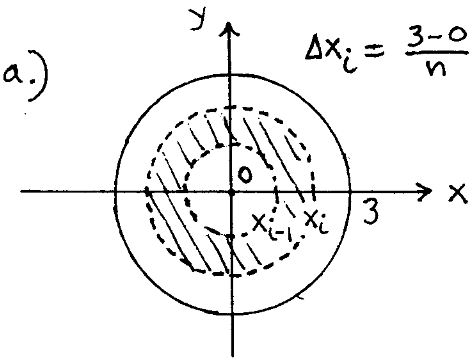
2.)



a.) $$Vol = 2\pi \int_{-1}^{1} (4-x)\sqrt{1-x^2}\,dx$$

b.) $$Vol = \pi \int_0^1 \left(4+\sqrt{1-y^2}\right)^2 dy + \pi \int_0^1 \left(4-\sqrt{1-y^2}\right)^2 dy$$

3.) a.)



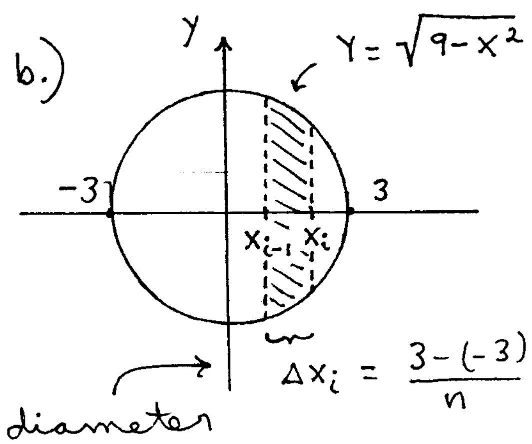
Estimate for mass of "strip" is

$$\underbrace{(2\pi x_i)(\Delta x_i)}_{\text{area}} \cdot \underbrace{(x_i^3)}_{\text{density}}$$

so that total mass is

$$Mass = \int_0^3 2\pi x\,(x^3)\,dx = 2\pi \frac{x^5}{5}\Big|_0^3 = \frac{486}{5}\pi \text{ kg.}$$

b.)



Estimate for mass of "strip" in right semi-circle is

$\underbrace{(\Delta x_i)(2\sqrt{9-x_i^2})}_{\text{area}} \cdot \underbrace{(x_i^3)}_{\text{density}}$ so that the total mass of <u>both</u> semi-circles

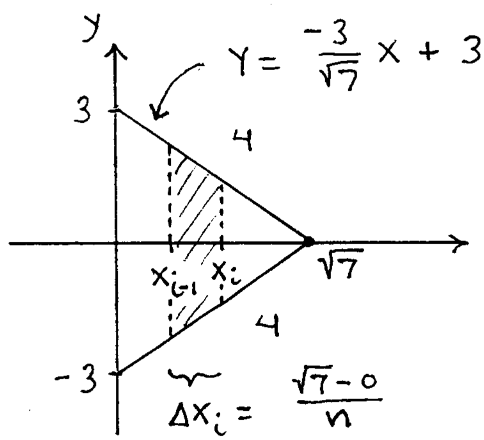
is $\text{Mass} = \underline{\underline{2}} \int_0^3 2x^3\sqrt{9-x^2}\,dx = 4\int_0^3 x^3\sqrt{9-x^2}\,dx$

$\Big($ let $u = x^2$, $dv = x\sqrt{9-x^2}\,dx$
$du = 2x\,dx$, $v = \frac{-1}{3}(9-x^2)^{3/2}$ $\Big)$

$= 4\left\{ \frac{-1}{3}x^2(9-x^2)^{3/2}\Big|_0^3 + \frac{2}{3}\int_0^3 x(9-x^2)^{3/2}\,dx \right\}$

$= \frac{8}{3}\cdot\frac{-1}{5}(9-x^2)^{5/2}\Big|_0^3 = \frac{648}{5}$ kg.

4.)



Estimate for mass of "strip" is

$\underbrace{(\Delta x_i)2\left(\frac{-3}{\sqrt{7}}x_i+3\right)}_{\text{area}} \cdot \underbrace{\frac{75}{\frac{1}{2}(6)\sqrt{7}}}_{\text{density}}$ kg.

and estimate for velocity of "strip" is $(2\pi x_i)(10)$ ft./min. so that estimate for kinetic energy of "strip" is

$\frac{1}{2}mv^2 = \frac{1}{2}(\Delta x_i)2\left(\frac{-3}{\sqrt{7}}x_i+3\right)\cdot\frac{25}{\sqrt{7}}\cdot(20\pi x_i)^2$

$= \frac{10{,}000}{\sqrt{7}}\pi^2\left(3x_i^2 - \frac{3}{\sqrt{7}}x_i^3\right)$;

thus the total kinetic energy is

$$K.E. = \frac{10{,}000}{\sqrt{7}} \pi^2 \int_0^{\sqrt{7}} \left(3x^2 - \frac{3}{\sqrt{7}} x^3\right) dx$$

$$= \frac{10{,}000}{\sqrt{7}} \pi^2 \left(x^3 - \frac{3}{\sqrt{7}} \frac{x^4}{4}\right) \Big|_0^{\sqrt{7}}$$

$$= \frac{10{,}000}{\sqrt{7}} \pi^2 \left((\sqrt{7})^3 - \frac{3}{4} (\sqrt{7})^3\right)$$

$$= 17{,}500 \pi^2 \quad \frac{kg. \ ft.^2}{min.^2}$$