ESP
Kouba
Worksheet 10 Solutions

1.) a.) nth term test: $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n\cdot -8} = e^{-8} \neq 0$

so series diverges

b.) limit comparison: $\lim_{n\to\infty} \dfrac{\frac{n+30}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \dfrac{n^{5/2}+30n^{3/2}}{n^{5/2}}$

$= \lim_{n\to\infty} \left(1+\frac{30}{n}\right) = 1$ so series converges since $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges.

c.) limit comparison: $\lim_{n\to\infty} \dfrac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$= \lim_{n\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$ so series diverges since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

d.) absolute ratio test: $\lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \dfrac{10^{2(n+1)}}{(3(n+1))!} \cdot \dfrac{(3n)!}{10^{2n}}$

$= \lim_{n\to\infty} \dfrac{10^2}{(3n+3)(3n+2)(3n+1)} = 0 < 1$ so series converges.

e.) limit comparison: $\lim_{n\to\infty} \dfrac{\cancel{\left(\frac{1}{10}\right)^n} \cdot \frac{n^2+1}{n^2+7}}{\cancel{\left(\frac{1}{10}\right)^n}} = \lim_{n\to\infty} \dfrac{1+\frac{1}{n^2}}{1+\frac{7}{n^2}} = 1$

so series converges absolutely since $\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$ converges.

f.) absolute root test : $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^{\frac{1}{\sqrt{n}}}}{4}$

$= \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right)^{n\cdot\frac{1}{n\sqrt{n}}} = \frac{1}{4} e^0 = \frac{1}{4} < 1$ so

series converges.

g.) $0 \le \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ converges

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges so original series

converges by absolute convergence test.

h.) nth term test : $\lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{n\to\infty} \frac{\cos\frac{1}{n}\cdot\cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} = \cos 0 = 1 \neq 0$ so

series diverges.

i.) integral test : $f(x) = \frac{1}{x\sqrt{x^2-1}}$ is +, cont.,

and ↓ with $\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{B\to\infty} \operatorname{arcsec} x \Big|_2^B$

$= \lim_{B\to\infty} (\operatorname{arcsec} B - \operatorname{arcsec} 2) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ so

series converges.

j.) sequence of partial sums :

$S_1 = \frac{1}{\ln 3} - \frac{1}{\ln 4}$

$S_2 = \left(\frac{1}{\ln 3} - \cancel{\frac{1}{\ln 4}}\right) + \left(\cancel{\frac{1}{\ln 4}} - \frac{1}{\ln 5}\right)$

$S_3 = \left(\frac{1}{\ln 3} - \cancel{\frac{1}{\ln 4}}\right) + \left(\cancel{\frac{1}{\ln 4}} - \cancel{\frac{1}{\ln 5}}\right) + \left(\cancel{\frac{1}{\ln 5}} - \frac{1}{\ln 6}\right)$

$\vdots$

$$s_n = \frac{1}{\ln 3} - \frac{1}{\ln(n+3)} \quad \text{and}$$

$$\sum_{n=3}^{\infty}\left[\frac{1}{\ln n} - \frac{1}{\ln(n+1)}\right] = \lim_{n\to\infty} s_n = \frac{1}{\ln 3} \quad \text{so}$$

series converges.

k.) integral test: $f(x) = \dfrac{\arctan x}{1+x^2}$ is +, cont, and ↓ for $x \geq 1$ and

$$\int_1^{\infty} \frac{\arctan x}{1+x^2}\,dx = \lim_{B\to\infty} \frac{1}{2}(\arctan x)^2\Big|_1^B$$

$$= \lim_{B\to\infty} \frac{1}{2}(\arctan B)^2 - \frac{1}{2}(\arctan 1)^2 = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{3\pi^2}{32}$$

so series converges.

2.) a.) $\displaystyle \lim_{n\to\infty} \frac{3^{n+1}|x|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{3^n|x|^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 \cdot 3|x| = 3|x| < 1$

$\Rightarrow |x| < \frac{1}{3} \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$; if $x = \frac{1}{3}$ then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$) and if $x = -\frac{1}{3}$ then $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges (alt. series) so power series converges for $-\frac{1}{3} \leq x \leq \frac{1}{3}$.

b.) $\displaystyle \lim_{n\to\infty} \frac{|2x|^{n+2}}{(n+1)!} \cdot \frac{n!}{|2x|^{n+1}} = \lim_{n\to\infty} \frac{|2x|}{n+1} = 0 < 1$ for all values of x, so series converges for all values of x

c.) $\displaystyle \lim_{n\to\infty} \frac{|x-5|^{n+1}}{|x-5|^n} = \lim_{n\to\infty} |x-5| = |x-5| < 1 \Rightarrow$

$-1 < x-5 < 1 \Rightarrow 4 < x < 6$; if $x = 4$ then $\displaystyle\sum_{n=0}^{\infty} (-1)^n$

diverges by nth-term test, and if $x=6$ then $\sum_{n=0}^{\infty}(1)^n$ diverges by nth-term test, so series converges for $4<x<6$.

d.) $$\lim_{n\to\infty} \frac{2^{n+2}\,|7-3x|^{2(n+1)}}{2^{n+1}\,|7-3x|^{2n}} = \lim_{n\to\infty} 2|7-3x|^2$$

$$= 2|7-3x|^2 < 1 \Rightarrow |7-3x| < \frac{1}{\sqrt{2}} \Rightarrow \frac{-1}{\sqrt{2}} < 7-3x < \frac{1}{\sqrt{2}} \Rightarrow$$

$$\frac{-1}{\sqrt{2}} - 7 < -3x < \frac{1}{\sqrt{2}} - 7 \Rightarrow \frac{7-\frac{1}{\sqrt{2}}}{3} < x < \frac{7+\frac{1}{\sqrt{2}}}{3} \;;$$

if $x = \frac{1}{3}\left(7-\frac{1}{\sqrt{2}}\right)$ then $\sum_{n=0}^{\infty} 2$ diverges, and if $x = \frac{1}{3}\left(7+\frac{1}{\sqrt{2}}\right)$ then $\sum_{n=0}^{\infty} 2$ diverges, so series converges for $\frac{1}{3}\left(7-\frac{1}{\sqrt{2}}\right) < x < \frac{1}{3}\left(7+\frac{1}{\sqrt{2}}\right)$.

e.) $$\lim_{n\to\infty} \frac{(n+2)!\,|x|^{2(n+1)+1}}{(n+1)!\,|x|^{2n+1}} = \lim_{n\to\infty} (n+2)|x|^2$$

$$= \begin{cases} \infty & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$ so series converges only for $x=0$.

f.) $$\lim_{n\to\infty} \frac{\ln(n+1)\,|2-x|^{n+1}}{\ln n\,|2-x|^n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} |2-x|$$

$$= \lim_{n\to\infty} \frac{n}{n+1}|2-x| = |2-x| < 1 \Rightarrow -1 < 2-x < 1 \Rightarrow$$

$-3 < -x < -1 \Rightarrow 1 < x < 3$; if $x=1$ then $\sum_{n=2}^{\infty} \ln n$ diverges (nth-term test), and if $x=3$ then $\sum_{n=2}^{\infty} (-1)^n \ln n$ diverges (nth-term test) so series converges for $1 < x < 3$.

g.) $\lim\limits_{n\to\infty} \dfrac{((n+1)^2|x|)^{n+1}}{(n^2|x|)^n} = \lim\limits_{n\to\infty} \dfrac{(n+1)^{2n+2}\,|x|^{n+1}}{n^{2n}\;|x|^n}$

$= \lim\limits_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^2 (n+1)^2|x| = \begin{cases} \infty & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

so series converges only for $x = 0$.

h.) $\lim\limits_{n\to\infty} \dfrac{(n+1)^2+1}{(n+1)^3-1}\,(9)^{n+1}|x-3|^{2(n+1)} \cdot \dfrac{n^3-1}{n^2+1} \cdot (9)^n\,|x-3|^{2n}$

$= \lim\limits_{n\to\infty} \dfrac{(n+1)^2+1}{n^2+1} \cdot \dfrac{n^3-1}{(n+1)^3-1} \cdot (9)|x-3|^2 = 9|x-3|^2 < 1 \Rightarrow$

$|x-3|^2 < \frac{1}{9} \Rightarrow |x-3| < \frac{1}{3} \Rightarrow -\frac{1}{3} < x-3 < \frac{1}{3} \Rightarrow$

$\frac{8}{3} < x < \frac{10}{3}$; if $x = \frac{\pm 8}{3}$ then $\sum\limits_{n=2}^{\infty} \dfrac{n^2+1}{n^3-1}$ diverges (limit comparison with $\frac{1}{n}$) so series converges for $\frac{8}{3} < x < \frac{10}{3}$.

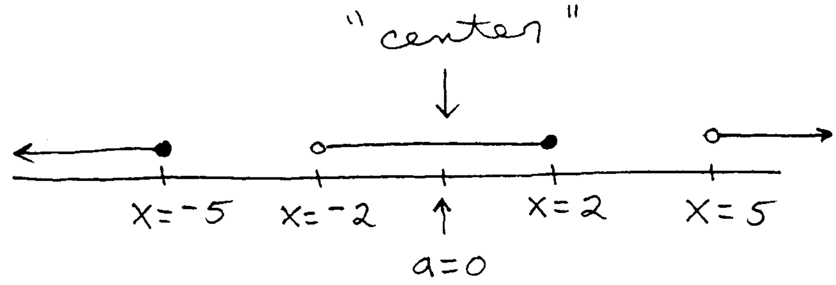
i.) $\lim\limits_{n\to\infty} \dfrac{((n+1)!)^2\,|x|^{n+1}}{(n+1)^{2(n+1)}} \cdot \dfrac{n^{2n}}{(n!)^2|x|^n} = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^{2n}|x|$

$= \lim\limits_{n\to\infty} \dfrac{1}{\left(1+\frac{1}{n}\right)^{n\cdot 2}}|x| = \dfrac{1}{e^2}|x| < 1 \Rightarrow |x| < e^2 \Rightarrow$

$-e^2 < x < e^2$; if $x = \pm e^2$ then $\sum\limits_{n=1}^{\infty} \dfrac{(n!)^2}{n^{2n}} e^{2n}$ and $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{(n!)^2}{n^{2n}} e^{2n}$ diverge since

$\lim\limits_{n\to\infty} \dfrac{(n!)^2 e^{2n}}{n^{2n}} \neq 0$ (nth term test), so series converges for $-e^2 < x < e^2$.

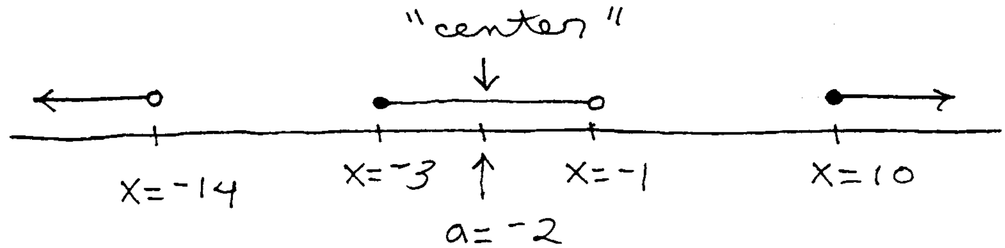
3.)



a.) converge : $-2 < x < 2$

b.) diverge : $x < -5$, $x > 5$

4.)



a.) converge : $-3 < x < -1$

b.) diverge : $x < -14$, $x > 10$

5.) assume $\sum_{n=1}^{\infty} a_n^2 = L < \infty$ and $\sum_{n=1}^{\infty} b_n^2 = M < \infty$.

a.) $\sum_{n=1}^{\infty} (3a_n^2 - 2b_n^2) = 3 \sum_{n=1}^{\infty} a_n^2 - 2 \sum_{n=1}^{\infty} b_n^2$

$= 3L - 2M$ is finite.

b.) $0 \leq (a_n - b_n)^2 \Rightarrow 0 \leq a_n^2 - 2a_n b_n + b_n^2 \Rightarrow$

$2 a_n b_n \leq a_n^2 + b_n^2 \Rightarrow 0 \leq a_n b_n \leq \frac{1}{2}(a_n^2 + b_n^2) \Rightarrow$

$\sum_{n=1}^{\infty} a_n b_n$ converges since

$\sum_{n=1}^{\infty} \frac{1}{2}(a_n^2 + b_n^2) = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right) = \frac{1}{2}(L + M) < \infty$

(comparison test)

c.) Part b.) states that **if** $\sum_{n=1}^{\infty} a_n^2$ and

$\sum_{n=1}^{\infty} b_n^2$ converge, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Thus, since $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2$ converge,

it follows that $\sum_{n=1}^{\infty} a_n \cdot \frac{1}{n}$ converges.

6.) a.) $$e^x + \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$
$$= 1 + 2x + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^5}{60} + \cdots$$

b.) $$x \cdot \cos\sqrt{x} = x\left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \frac{(\sqrt{x})^8}{8!} - \cdots\right]$$
$$= x - \frac{x^2}{2!} + \frac{x^3}{4!} - \frac{x^4}{6!} + \frac{x^5}{8!} - \cdots$$

c.) $$e^{-x}\cos 2x = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)$$
$$= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \cdots$$
$$- 2x^2 + 2x^3 - x^4 + \cdots$$
$$+ \frac{2}{3}x^4 - \cdots$$
$$= 1 - x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{7}{24}x^4 + \cdots$$

d.) $$\sin(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots$$

$$\begin{array}{r|l}
 & x^2 - x^3 + \frac{1}{2}x^4 - \frac{1}{6}x^5 - \frac{1}{8}x^6 + \cdots \\
\hline
1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots & x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10} - \\
 & \underline{x^2 + x^3 + \frac{1}{2}x^4 + \frac{1}{6}x^5 + \frac{1}{24}x^6 + \cdots} \\
 & -x^3 - \frac{1}{2}x^4 - \frac{1}{6}x^5 - \frac{5}{24}x^6 - \cdots \\
 & \underline{-x^3 - x^4 - \frac{1}{2}x^5 - \frac{1}{6}x^6 - \cdots}
\end{array}$$

$$
\begin{array}{r}
\frac{1}{2}x^4 + \frac{1}{3}x^5 - \frac{1}{24}x^6 + \cdots \\
\frac{1}{2}x^4 + \frac{1}{2}x^5 + \frac{1}{4}x^6 + \cdots \\
\hline
\frac{-1}{6}x^5 + \frac{-7}{24}x^6 + \cdots \\
\frac{-1}{6}x^5 + \frac{-1}{6}x^6 + \cdots \\
\hline
\frac{-1}{8}x^6 + \cdots
\end{array}
$$

so

$$\frac{\sin(x^2)}{e^x} = x^2 - x^3 + \frac{1}{2}x^4 - \frac{1}{6}x^5 - \frac{1}{8}x^6 + \cdots$$

7.) a.) $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots$

b.) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$ (integrate a.))

c.) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$ (substitute in a.))

d.) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$ (integrate c.))

e.) $\frac{2x}{1+x^2} = 2x \cdot \left(\frac{1}{1+x^2}\right) = 2x\left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right)$

$= 2x - 2x^3 + 2x^5 - 2x^7 + 2x^9 - \cdots$

f.) $\ln(1+x^2) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \frac{1}{5}x^{10} - \cdots$

(integrate e.) or substitute in b.))

g.) $\frac{1}{2+x} = \frac{\frac{1}{2}}{1+\frac{x}{2}} = \frac{1}{2} \cdot \frac{1}{1+\left(\frac{x}{2}\right)} = \frac{1}{2}\left[1 - \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 - \cdots\right]$

$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + \frac{1}{32}x^4 - \cdots$ (substitute in a.))

8.) a.) $e^{x^2}-1 = 1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\cdots -1 = x^2+\frac{1}{2}x^4+\frac{1}{6}x^6+\cdots$ and

$1-\frac{x^2}{2}-\cos x = 1-\frac{x^2}{2}-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right) = \frac{-x^4}{24}+\frac{x^6}{720}-\cdots$ so

$$(e^{x^2}-1)^2 = \left(x^2+\tfrac{1}{2}x^4+\tfrac{1}{6}x^6+\cdots\right)\left(x^2+\tfrac{1}{2}x^4+\tfrac{1}{6}x^6+\cdots\right)$$

$$= x^4+\tfrac{1}{2}x^6+\tfrac{1}{6}x^8+\cdots$$

$$+\tfrac{1}{2}x^6+\tfrac{1}{4}x^8+\cdots$$

$$+\tfrac{1}{6}x^8+\cdots \quad = x^4+x^6+\tfrac{7}{12}x^8+\cdots$$ then

$$\lim_{x\to 0}\frac{(e^{x^2}-1)^2}{1-\frac{x^2}{2}-\cos x} = \lim_{x\to 0}\frac{x^4+x^6+\frac{7}{12}x^8+\cdots}{-\frac{x^4}{24}+\frac{x^6}{720}-\cdots}$$

$$= \lim_{x\to 0}\frac{\cancel{x^4}\left(1+x^2+\frac{7}{12}x\ +\cdots\right)}{\cancel{x^4}\left(\frac{-1}{24}+\frac{1}{720}x^2-\cdots\right)} = \frac{1}{\frac{-1}{24}} = -24.$$

b.) $1-\cos(x^2) = 1-\left(1-\frac{x^4}{2!}+\frac{x^8}{4!}-\frac{x^{12}}{6!}+\cdots\right) = \frac{1}{2}x^4-\frac{1}{24}x^8-\frac{1}{720}x^{12}+\cdots$

and $x-\sin x = x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) = \frac{1}{6}x^3-\frac{1}{120}x^5+\frac{1}{5040}x^7-\cdots$

then

$$\lim_{x\to 0}\frac{(1-\cos x^2)^3}{(x-\sin x)^4} = \lim_{x\to 0}\frac{\left(\frac{1}{2}x^4-\frac{1}{24}x^8-\frac{1}{720}x^{12}+\cdots\right)^3}{\left(\frac{1}{6}x^3-\frac{1}{120}x^5+\frac{1}{5040}x^7\cdots\right)^4}$$

$$= \lim_{x\to 0}\frac{\cancel{(x^4)^3}\left(\frac{1}{2}-\frac{1}{24}x^4-\frac{1}{720}x^8+\cdots\right)^3}{\cancel{(x^3)^4}\left(\frac{1}{6}-\frac{1}{120}x^2+\frac{1}{5040}x^4-\cdots\right)^4} = \frac{\left(\frac{1}{2}\right)^3}{\left(\frac{1}{6}\right)^4} = 162$$