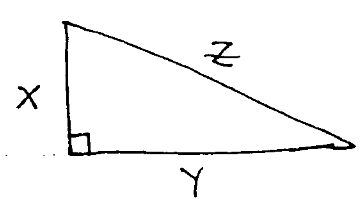
ESP
Kouba
Worksheet 4 Solutions

1.) a.) $$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{(x^2+y^2)}} = 0 .$$

b.) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{5x^4+3y^4} = \text{"}\frac{0}{0}\text{"}$; along the path $y=x$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{5x^4+3y^4} = \lim_{(x,y)\to(0,0)} \frac{2x^2}{8x^4}$ $\displaystyle = \lim_{(x,y)\to(0,0)} \frac{1}{4x^2} = +\infty$; along the path $y=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{5x^4+3y^4} = \lim_{(x,y)\to(0,0)} \frac{0}{5x^4} = \lim_{(x,y)\to(0,0)} 0 = 0$, so that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{5x^4+3y^4}$ does not exist.

c.) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^{3/2}-xy}{x^{3/2}+y^3} = \text{"}\frac{0}{0}\text{"}$; along the path $x=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{0}{y^3} = \lim_{(x,y)\to(0,0)} 0 = 0$; along the path $y=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^{3/2}}{x^{3/2}} = \lim_{(x,y)\to(0,0)} 1 = 1$, so that $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x^{3/2}-xy}{x^{3/2}+y^3}\right)$ does not exist.

2.) a.) $z=\sqrt{x^2+y^2}$ then



$$|\Delta z| \approx |dz| = |\, z_x \cdot \Delta x + z_y \cdot \Delta y \,|$$

$$\le \frac{x}{\sqrt{x^2+y^2}} \cdot |\Delta x| + \frac{y}{\sqrt{x^2+y^2}} \cdot |\Delta y|$$

$x=3$, $y=4$,
$\Delta x = \Delta y = 0.02$

$$= \frac{3}{5}(0.02) + \frac{4}{5}(0.02) = 0.028 \text{ cm}.$$

b.) $A = \frac{1}{2}XY$ then

$$|\Delta A| \approx |dA| = |A_X \cdot \Delta X + A_Y \cdot \Delta Y|$$

$$\leq \frac{1}{2}Y \cdot |\Delta X| + \frac{1}{2}X \cdot |\Delta Y| = 2(0.02) + \frac{3}{2}(0.02) = 0.07 \text{ cm}^2.$$

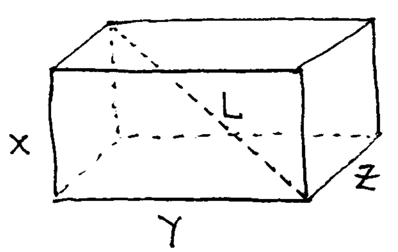
3.) $w = r^2 + 3sv + 2p^3$, $r = 1$, $\Delta r = 0.02$, $s = 2$, $\Delta s = -0.01$, $v = 4$, $\Delta v = 0.01$, $p = 3$, $\Delta p = -0.03$ so

$$\Delta w \approx dw = w_r \cdot \Delta r + w_s \cdot \Delta s + w_v \cdot \Delta v + w_p \cdot \Delta p$$

$$= 2r \cdot \Delta r + 3v \cdot \Delta s + 3s \cdot \Delta v + 6p^2 \cdot \Delta p$$

$$= 2(0.02) + 12(-0.01) + 6(0.01) + 54(-0.03) = -1.64$$

4.)



$X = 9$, $Y = 12$, $Z = 8$
$|\Delta X| = 0.01$, $|\Delta Y| = 0.02$, $|\Delta Z| = 0.03$

$L = \sqrt{X^2 + Y^2 + Z^2}$ then

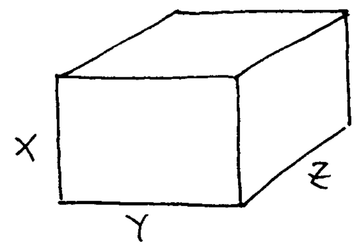
$$|\Delta L| \approx |dL| = |L_X \cdot \Delta X + L_Y \cdot \Delta Y + L_Z \cdot \Delta Z|$$

$$\leq \frac{X}{\sqrt{X^2+Y^2+Z^2}}|\Delta X| + \frac{Y}{\sqrt{X^2+Y^2+Z^2}}|\Delta Y| + \frac{Z}{\sqrt{X^2+Y^2+Z^2}}|\Delta Z|$$

$$= \frac{9}{17}(0.01) + \frac{12}{17}(0.02) + \frac{8}{17}(0.03) \approx 0.0335 \text{ ft}.$$

5.) a.) $V = (1.02)(3.01)(4.2) = 12.89484 \text{ cm}^3$

b.)



$x = 1, \ y = 3, \ z = 4$

$\Delta x = 0.02, \ \Delta y = 0.01, \ \Delta z = 0.2$

$V = xyz$ so

$$V(x+\Delta x, y+\Delta y, z+\Delta z) - V(x,y,z) = \Delta V \approx dV$$
$$= V_x \cdot \Delta x + V_y \cdot \Delta y + V_z \cdot \Delta z$$
$$= yz \cdot \Delta x + xz \cdot \Delta y + xy \cdot \Delta z \ , \text{ i.e.,}$$

$$V(1.02, 3.01, 4.2) - V(1,3,4) \approx 12(0.02) + 4(0.01) + 3(0.2) \quad \text{or}$$

$$V(1.02, 3.01, 4.2) \approx 12 + 0.88 = 12.88$$

6.) $S = \dfrac{A}{A-W}$, $A = 12, \ W = 5, \ |\Delta A| = \frac{1}{32}, \ |\Delta W| = \frac{1}{16}$ so

$$|\Delta S| \approx |dS| = |S_A \cdot \Delta A + S_W \cdot \Delta W|$$
$$\leq \frac{W}{(A-W)^2} \cdot |\Delta A| + \frac{A}{(A-W)^2} \cdot |\Delta W|$$
$$= \frac{5}{49}\left(\frac{1}{32}\right) + \frac{12}{49}\left(\frac{1}{16}\right) = \frac{29}{1568} \ .$$

7.) $$\frac{dw}{dt} = w_u \cdot \frac{du}{dt} + w_v \cdot \frac{dv}{dt}$$
$$= \frac{3}{3u+v^2} \cdot (-2e^{-2t}) + \frac{2v}{3u+v^2} \cdot (3t^2 - 2t)$$

$$= \frac{-6e^{-2t}}{3e^{-2t}+(t^3-t^2)^2} + \frac{2(t^3-t^2)(3t^2-2t)}{3e^{-2t}+(t^3-t^2)^2}$$

8.) $u = 3t^2 - s$ and $w = f(u)$ so

$$\frac{\partial w}{\partial t} = f'(u)\cdot u_t$$

$$= \sin(3t^2-5)\cdot 6t \quad \text{and}$$

$$\frac{\partial w}{\partial s} = f'(u)\cdot u_s = \sin(3t^2-5)\cdot(-1)$$

9.) assume $z = f(x,y)$ and $xy^2 + z^2 + \cos(xyz) = 4$ so partial differentiation w.r.t. x gives

$$y^2 + 2z\cdot z_x - \sin(xyz)\cdot\{xy\cdot z_x + y\cdot z\} = 0 \rightarrow$$

$$[2z - xy\sin(xyz)]\,z_x = yz\cdot\sin(xyz) - y^2 \rightarrow$$

$$z_x = \frac{yz\cdot\sin(xyz) - y^2}{2z - xy\sin(xyz)}.$$

10.) $u = ax + by$ and $w = f(u)$ so

$$\frac{\partial w}{\partial y} = f'(u)\cdot u_y = b\cdot f'(u) \quad \text{and}$$

$$\frac{\partial w}{\partial x} = f'(u)\cdot u_x = a\cdot f'(u) \; ; \quad \text{then}$$

$$a\left(\frac{\partial w}{\partial y}\right) = ab\,f'(u) = b\left(\frac{\partial w}{\partial x}\right).$$

11.) $u = xy$ and $z = x \cdot f(u)$ so

$z_x = x \cdot f'(u) \cdot u_x + 1 \cdot f(u) = xy f'(u) + f(u)$ and

$z_y = x \cdot f'(u) \cdot u_y = x^2 f'(u)$; then

$$x \cdot z_x - y \cdot z_y = x(xy f'(u) + f(u)) - y(x^2 f'(u))$$
$$= x \cdot f(u)$$
$$= z \; .$$

12.) $u = f(x+at) + g(x-at)$ so

$$\frac{\partial u}{\partial x} = f'(x+at) \cdot 1 + g'(x-at) \cdot 1 ,$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+at) \cdot 1 + g''(x-at) \cdot 1 ,$$

$$\frac{\partial u}{\partial t} = f'(x+at) \cdot a + g'(x-at) \cdot -a ,$$

$$\frac{\partial^2 u}{\partial t^2} = f''(x+at) \cdot a^2 + g''(x-at) \cdot a^2 ;$$

13.) a.) $z_X = 3X^2 - 3Y^2 = 3(X-Y)(X+Y) = 0 \rightarrow Y = X$ or $Y = -X$,

$z_Y = -6XY + 6Y = 6Y(1-X) = 0 \rightarrow Y = 0$ or $X = 1$

so critical points are $(1,1)$, $(1,-1)$, $(0,0)$;

$z_{XX} = 6X$, $z_{YY} = -6X+6$, $z_{XY} = -6Y$.

$(0,0)$: $D = z_{XX} z_{YY} - (z_{XY})^2 = (0)(6) - (0)^2 = 0$ so this test is inconclusive ;

$(1,1)$: $D = (6)(0) - (-6)^2 = -36 < 0$ so $(1,1)$ determines a saddle point ;

$(1,-1)$: $D = (6)(0) - (6)^2 = -36 < 0$ so $(1,-1)$ determines a saddle point .

b.) $z_X = 6X - 6Y + 12 = 0 \rightarrow X - Y + 2 = 0$
$z_Y = -6X + 2Y - 16 = 0 \rightarrow -3X + Y - 8 = 0$
$\Big\} \begin{matrix} Y = X+2 \\ Y = 3X+8 \end{matrix} \Big\}$

$X + 2 = 3X + 8 \rightarrow -6 = 2X \rightarrow X = -3$, $Y = -1$;

$z_{XX} = 6$, $z_{YY} = 2$, $z_{XY} = -6$.

$(-3,-1)$: $D = z_{XX} \cdot z_{YY} - (z_{XY})^2 = (6)(2) - (-6)^2 = -24 < 0$ so $(-3,-1)$ determines a saddle point .

c.) $z_X = 2X - \frac{1}{XY} \cdot Y = 2X - \frac{1}{X} = 0 \rightarrow 2X^2 = 1 \rightarrow X = \pm \frac{1}{\sqrt{2}}$,

$z_Y = \frac{-1}{XY} \cdot X + 2Y = 2Y - \frac{1}{Y} = 0 \rightarrow 2Y^2 = 1 \rightarrow Y = \pm \frac{1}{\sqrt{2}}$,

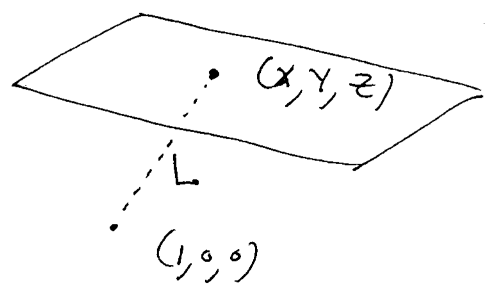
so critical points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$;

$z_{XX} = 2 + \frac{1}{X^2}$, $z_{YY} = 2 + \frac{1}{Y^2}$, $z_{XY} = 0$.

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$: $D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (4)(4) - (0)^2 = 16 > 0$ and $z_{xx} = 4 > 0$ so $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ determines a minimum value; similarly, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ also determine minimum values.

14.) Point $(1,0,0)$ is on plane $2x + 3y - z = 2$, so find minimum distance from $(1,0,0)$ to plane $2x + 3y - z = 4$

or $z = 2x + 3y - 4$ $\rightsquigarrow$



$$L = \sqrt{(x-1)^2 + (y-0)^2 + (z-0)^2}$$
$$= \sqrt{(x-1)^2 + y^2 + (2x+3y-4)^2},$$

$$L_x = \tfrac{1}{2}(\sim)^{-1/2} \cdot \left[2(x-1) + 2(2x+3y-4)\cdot 2\right] = 0 \rightarrow$$
$$x - 1 + 4x + 6y - 8 = \boxed{5x + 6y - 9 = 0},$$

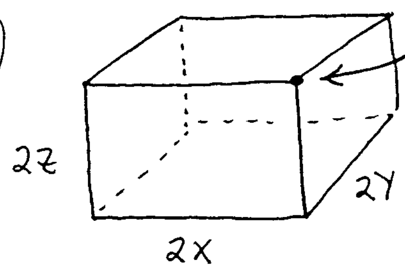
$$L_y = \tfrac{1}{2}(\sim)^{-1/2}\left[2y + 2(2x+3y-4)\cdot 3\right] = 0 \rightarrow$$
$$y + 6x + 9y - 12 = 6x + 10y - 12 = 0 \rightarrow \boxed{3x + 5y - 6 = 0}$$

$$\left.\begin{matrix} 5x + 6y - 9 = 0 \\ 3x + 5y - 6 = 0 \end{matrix}\right\} \quad \left.\begin{matrix} 15x + 18y - 27 = 0 \\ 15x + 25y - 30 = 0 \end{matrix}\right\} \quad 7y = 3 \rightarrow$$

$y = \frac{3}{7}$ and $x = \frac{9}{7}$ and $z = \frac{-1}{7}$ so minimum distance is

$$L = \sqrt{\left(\frac{9}{7}-1\right)^2+\left(\frac{3}{7}-0\right)^2+\left(-\frac{1}{7}-0\right)^2} = \frac{\sqrt{14}}{7}.$$

15.)



pt. is (x,y,z) if we assume origin is in the center of the box and (x,y,z) is the corner of the box in octant 1, which is also on the ellipsoid $16x^2+4y^2+9z^2=144$; maximize volume

$$V=(2x)(2y)(2z) = 8xy\cdot\frac{1}{3}\sqrt{144-16x^2-4y^2} \rightarrow$$

$$V_x = \frac{8}{3}xy\cdot\frac{1}{2}\left(144-16x^2-4y^2\right)^{-1/2}\cdot(-32x) + \frac{8}{3}y\cdot\sqrt{144-16x^2-4y^2} = 0$$

$$\Rightarrow \cdots \Rightarrow \quad 8x^2+y^2=36 \quad \text{and}$$

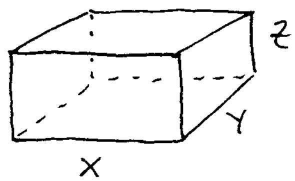
$$V_y = \frac{8}{3}xy\cdot\frac{1}{2}\left(144-16x^2-4y^2\right)^{-1/2}\cdot(-8y) + \frac{8}{3}x\sqrt{144-16x^2-4y^2} = 0$$

$$\Rightarrow \cdots \Rightarrow \quad 2x^2+y^2=18$$

$\Rightarrow \cdots \Rightarrow \quad x=\sqrt{3},\ y=2\sqrt{3},\ z=\frac{4}{3}\sqrt{3}$ so dimensions of box are

$2\sqrt{3}$, $4\sqrt{3}$, and $\frac{8}{3}\sqrt{3}$.

16.)



$xyz = 8 \rightarrow z = \frac{8}{xy}$

minimize surface area

$$S = 2xy + 2xz + 2yz$$

$$= 2xy + 2x\left(\frac{8}{xy}\right) + 2y\left(\frac{8}{xy}\right) = 2xy + \frac{16}{y} + \frac{16}{x} \rightarrow$$

$$\left.\begin{array}{l} S_x = 2y - \frac{16}{x^2} = 0 \rightarrow \quad y = \frac{8}{x^2} \\ S_y = 2x - \frac{16}{y^2} = 0 \rightarrow \quad x = \frac{8}{y^2} \end{array}\right\} \quad y = \frac{8}{\left(\frac{8}{y^2}\right)^2} = \frac{1}{8}y^4 \rightarrow$$

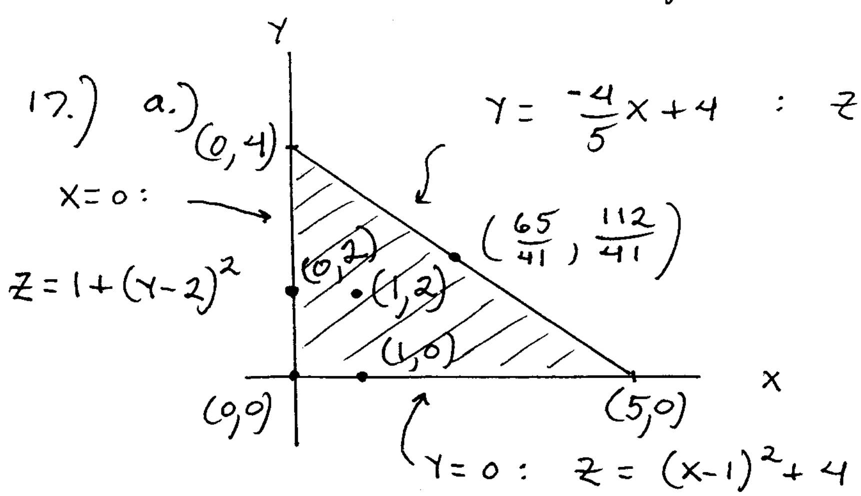
$8y - y^4 = y(8 - y^3) = 0 \rightarrow y = 0$ (NO!) or $y = 2 \rightarrow$

$x = 2$, and $z = 2$ so min. surface area is

$$S = 24 \text{ ft.}^2$$

17.) a.)

$y = \frac{-4}{5}x + 4 \quad : \quad z = (x-1)^2 + (2 - \frac{4}{5}x)^2$



$$z = (x-1)^2 + (y-2)^2 \rightarrow$$

$$z_x = 2(x-1) = 0 \rightarrow x = 1 \quad \text{and}$$

$z_y = 2(y-2) = 0 \rightarrow y = 2$ so critical point is $\underline{\underline{(1,2)}}$;

along $x=0$: $z' = 2(y-2) = 0 \rightarrow y = 2$ so $\underline{\underline{(0,2)}}$ is critical pt. on edge ;

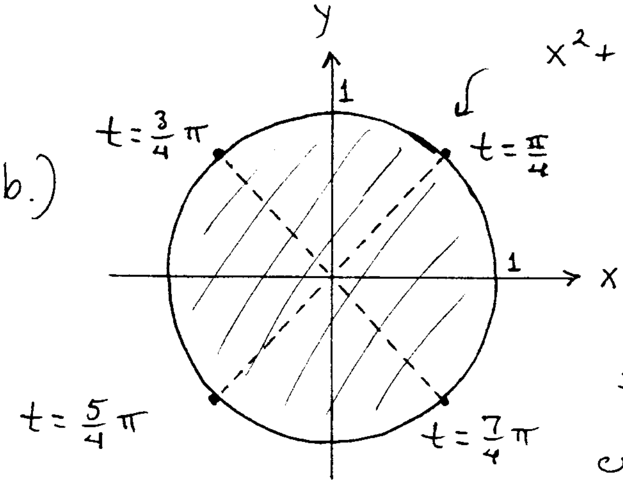
along $y=0$: $z' = 2(x-1) = 0 \rightarrow x = 1$ so $\underline{\underline{(1,0)}}$ is critical pt. on edge ;

along $y = -\frac{4}{5}x + 4$: $z' = 2(x-1) + 2(2 - \frac{4}{5}x) \cdot -\frac{4}{5} = 0 \rightarrow$ $x = \frac{65}{41}$ so $\left(\frac{65}{41}, \frac{112}{41}\right)$ is critical pt. on edge;

determine maximum and minimum z-values for following ordered pairs :

$(0,0)$	:	$z = 5$	
$(0,2)$	:	$z = 1$	
$(0,4)$	:	$z = 5$	
$(1,0)$	:	$z = 4$	
$(1,2)$	:	$z = 0$	$\leftarrow$ min.
$(5,0)$	:	$z = 20$	$\leftarrow$ max.
$\left(\frac{65}{41}, \frac{112}{41}\right)$	:	$z = .878$	

b.)



$x^2 + y^2 = 1$ or

$x = \cos t$

$y = \sin t$

$z = xy \rightarrow$

$z_x = y = 0, \; z_y = x = 0$ so critical pt. is $(0,0)$;

on circle:

$z = xy = \cos t \cdot \sin t \rightarrow$

$z' = \cos t \cdot \cos t - \sin t \cdot \sin t$

$= \cos^2 t - \sin^2 t$

$= (\cos t - \sin t)(\cos t + \sin t) = 0 \rightarrow$

$\cos t = \sin t$ or $\cos t = -\sin t \rightarrow$

$t = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ or $t = \frac{3}{4}\pi$ or $\frac{7}{4}\pi$ so

check z-values:

$(0,0)$: $z = 0$

$t = \frac{\pi}{4} \rightarrow \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$: $z = \frac{1}{2} \leftarrow$ max.

$t = \frac{3}{4}\pi \rightarrow \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$: $z = \frac{-1}{2} \leftarrow$ min.

$t = \frac{5}{4}\pi \rightarrow \left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$: $z = \frac{1}{2} \leftarrow$ max.

$t = \frac{7}{4}\pi \rightarrow \left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$: $z = \frac{-1}{2} \leftarrow$ min.