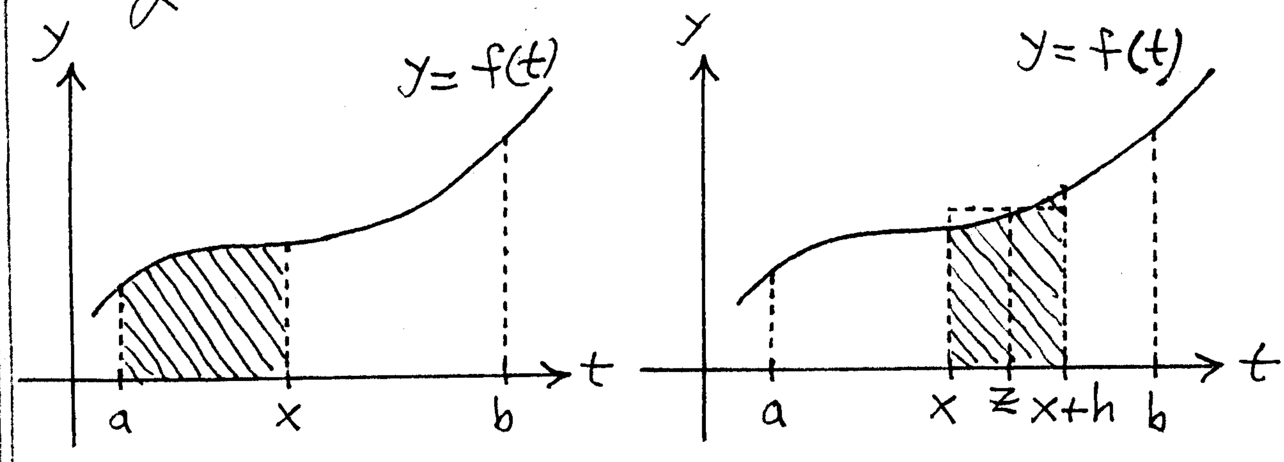
Math 16B
Kouba
Why is area an Antiderivative?



Let $A(x)= \int_a^x f(t)\,dt$ represent the area of the region under the graph of $y = f(t)$ and above the t-axis from $t=a$ to $t=x$. Note that

$$A(a) = \int_a^a f(t)\,dt = 0 \quad \text{and}$$

$$A(b) = \int_a^b f(t)\,dt \quad .$$

Find the derivative of $A(x)$:

$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$$

$$= \lim_{h\to 0} \frac{\text{"area of shaded strip, } x \text{ to } x+h\text{"}}{h}$$

$$= \lim_{h\to 0} \frac{\text{"area of rectangle formed by } z\text{"}}{h}$$

$$= \lim_{h\to 0} \frac{(\text{base})(\text{height})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\cdot f(z)}{\cancel{h}}$$

$$= \lim_{h\to 0} f(z)$$

$$= f(x) \quad . \quad \text{Thus, } A'(x) = f(x)$$

so $A(x)$ is an antiderivative for $f(x)$. Let

$$\int f(x)\,dx = F(x) + C$$

be the most general antiderivative for $f(x)$. It follows that

$$A(x) = F(x) + C \quad ;$$

and $A(a) = F(a) + C \to 0 = F(a) + C \to C = -F(a)$, $A(b) = F(b) + C = F(b) - F(a)$, i.e.,

$$\int_a^b f(t)\,dt = F(b) - F(a) \quad \text{or}$$

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)}$$

(Fundamental Theorem of Calculus)