Section 6.3

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2.) $\dfrac{3x+11}{x^2-2x-3} = \dfrac{3x+11}{(x-3)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x+1} \rightarrow$

$3x+11 = A(x+1) + B(x-3) \rightarrow$

$x=3: \quad 20 = 4A \rightarrow A=5$

$x=-1: \quad 8 = -4B \rightarrow B=-2$

6.) $\dfrac{7x+5}{6(2x^2+3x+1)} = \dfrac{7x+5}{6(2x+1)(x+1)} = \dfrac{1}{6}\left[\dfrac{A}{2x+1} + \dfrac{B}{x+1}\right] \rightarrow$

$7x+5 = A(x+1) + B(2x+1) \rightarrow$

$x=-1: \quad -2 = -B \rightarrow B=2$

$x=\frac{-1}{2}: \quad \frac{3}{2} = \frac{1}{2}A \rightarrow A=3$

8.) $\dfrac{3x^2-x+1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2} \rightarrow$

$3x^2-x+1 = A(x+1)^2 + Bx(x+1) + Cx \rightarrow$

$x=0: \quad 1=A$

$x=-1: \quad 5 = -C \rightarrow C=-5$

$x=1: \quad 3 = 1(4) + B(2) - 5(1)$

$\rightarrow 2B = 4 \rightarrow B=2$

12.) $\dfrac{6x^2-5x}{x^3+6x^2+12x+8} = \dfrac{6x^2-5x}{(x+2)^3} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3} \rightarrow$

$6x^2-5x = A(x+2)^2 + B(x+2) + C \rightarrow$

$x=-2: \quad 34=C$

$x=0: \quad 0 = 4A+2B+34 \longrightarrow 44 = 2B+102 \rightarrow B=-29$

$x=-1: \quad 11 = A+B+34 \rightarrow 44 = 4A+4B+136$

and $A=6$

13.) $\int \left(\frac{1}{x^2-1}\right) dx = \int \frac{1}{(x-1)(x+1)} dx = \int \left[\frac{A}{x-1} + \frac{B}{x+1}\right] dx$

$\rightarrow \quad 1 = A(x+1) + B(x-1) \quad \rightarrow$

$x=1: \quad 1 = 2A \rightarrow A = 1/2$,

$x=-1: \quad 1 = -2B \rightarrow B = -1/2$

$= \int \left[\frac{1/2}{x-1} + \frac{-1/2}{x+1}\right] dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$

18.) $\int \frac{3}{x^2-3x} dx = \int \frac{3}{x(x-3)} dx = \int \left[\frac{A}{x} + \frac{B}{x-3}\right] dx$

$\rightarrow \quad 3 = A(x-3) + B(x) \quad \rightarrow$

$x=0: \quad 3 = -3A \rightarrow A = -1$,

$x=3: \quad 3 = 3B \rightarrow B = 1$

$= \int \left[\frac{-1}{x} + \frac{1}{x-3}\right] dx = -\ln|x| + \ln|x-3| + c$

20.) $\int \frac{5}{x^2+x-6}\,dx = \int\left[\frac{A}{x-2}+\frac{B}{x+3}\right]dx$

$\rightarrow A(x+3)+B(x-2)=5 \begin{cases} x=-3: & -5B=5 \rightarrow B=-1 \\ x=2: & 5A=5 \rightarrow A=1 \end{cases}$

$= \int\left[\frac{1}{x-2}+\frac{-1}{x+3}\right]dx = \ln|x-2|-\ln|x+3|+c$

26.) $\int \frac{3x^2-7x-2}{x^3-x}\,dx = \int\left[\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}\right]dx$

$\rightarrow A(x-1)(x+1)+Bx(x+1)+Cx(x-1)=3x^2-7x-2$

$\rightarrow x=1: \quad 2B=-6 \rightarrow B=-3$

$x=-1: \quad 2C=8 \rightarrow C=4$

$x=0: \quad -A=-2 \rightarrow A=2$

$= \int\left[\frac{2}{x}+\frac{-3}{x-1}+\frac{4}{x+1}\right]dx = 2\ln|x|-3\ln|x-1|+4\ln|x+1|+c$

29.) $\int \frac{4-3x}{(x-1)^2}\,dx = \int\left[\frac{A}{x-1}+\frac{B}{(x-1)^2}\right]dx$

$\rightarrow 4-3x = A(x-1)+B$

$\rightarrow x=1: \quad 1=B$

$x=0: \quad 4=-A+1 \rightarrow A=-3$

$= \int\left[\frac{-3}{x-1}+\frac{1}{(x-1)^2}\right]dx = \int\left[\frac{-3}{x-1}+(x-1)^{-2}\right]dx$

$= -3\ln|x-1| + \frac{(x-1)^{-1}}{-1} + c$

30.) $$\frac{x^4}{(x-1)^3} = \frac{x^4}{x^3-3x^2+3x-1} = x+3+\frac{6x^2-8x+3}{(x-1)^3}$$

since

$$\begin{array}{r} x + 3 \\ x^3-3x^2+3x-1 \overline{\big) \; x^4 \qquad\qquad\qquad} \\ \underline{x^4-3x^3+3x^2-x} \\ 3x^3-3x^2+x \\ \underline{3x^3-9x^2+9x-3} \\ 6x^2-8x+3 \end{array}$$

$$\int \frac{x^4}{(x-1)^3}\,dx = \int \left[x+3+\frac{6x^2-8x+3}{(x-1)^3}\right] dx$$

$$= \int \left[x+3+\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}\right] dx$$

$\rightarrow$ $A(x-1)^2+B(x-1)+C = 6x^2-8x+3$

$\rightarrow$ $x=1:\ C=1$

$x=2:\ A+B+1=11$

$x=0:\ A-B+1=3$

$\Big\}$ $2A+2=14 \rightarrow A=6$ and $B=4$

$$= \int \left[x+3+\frac{6}{x-1}+\frac{4}{(x-1)^2}+\frac{1}{(x-1)^3}\right] dx$$

$$= \tfrac{1}{2}x^2+3x+6\ln|x-1|-4(x-1)^{-1}-\tfrac{1}{2}(x-1)^{-2}+C.$$

35.) $$\int_1^5 \frac{x-1}{x^2(x+1)}\,dx = \int_1^5 \left[\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}\right] dx$$

$\rightarrow$ $Ax(x+1)+B(x+1)+Cx^2 = x-1$

$\rightarrow$ $x=0:\ B=-1$

$x=-1:\ C=-2$

$x=1:\ 2A-2-2=0 \rightarrow A=2$

$$= \int_1^5 \left[\frac{2}{x} + \frac{-1}{x^2} + \frac{-2}{x+1} \right] dx$$
$$= \left(2\ln|x| + \frac{1}{x} - 2\ln|x+1| \right) \Big|_1^5$$
$$= \left(2\ln 5 + \frac{1}{5} - 2\ln 6\right) - \left(2\overset{0}{\cancel{\ln 1}} + 1 - 2\ln 2\right)$$
$$= \frac{-4}{5} + \ln\frac{25}{9}$$

38.)

$$x^2-4 \overset{\displaystyle x}{\overline{\big)\, x^3 - 1}}$$
$$\underline{x^3 - 4x}$$
$$4x - 1$$

so that

$$\int_0^1 \frac{x^3-1}{x^2-4}\, dx = \int_0^1 \left[x + \frac{4x-1}{x^2-4} \right] dx = \int_0^1 \left[x + \frac{A}{x-2} + \frac{B}{x+2} \right] dx$$

$$\longrightarrow \quad 4x-1 = A(x+2) + B(x-2)$$
$$\longrightarrow \quad x=2:\ 7 = 4A \rightarrow A = 7/4$$
$$x=-2:\ -9 = -4B \rightarrow B = 9/4$$

$$= \int_0^1 \left[x + \frac{7/4}{x-2} + \frac{9/4}{x+2} \right] dx = \left(\frac{x^2}{2} + \frac{7}{4}\ln|x-2| + \frac{9}{4}\ln|x+2| \right) \Big|_0^1$$
$$= \left(\frac{1}{2} + \frac{7}{4}\overset{0}{\cancel{\ln 1}} + \frac{9}{4}\ln 3 \right) - \left(0 + \frac{7}{4}\ln 2 + \frac{9}{4}\ln 2 \right)$$
$$= \frac{1}{2} - 4\ln 2 + \frac{9}{4}\ln 3$$

42.) Area $= \displaystyle\int_{-1}^{2} \frac{-4}{x^2-x-6}\, dx = \int_{-1}^{2} \frac{-4}{(x-3)(x+2)}\, dx$

$$= \int_{-1}^{2} \left[\frac{A}{x-3} + \frac{B}{x+2} \right] dx \quad \longrightarrow$$

$$-4 = A(x+2) + B(x-3) \longrightarrow$$
$$x=3:\ -4 = 5A \rightarrow A = -4/5,$$
$$x=-2:\ -4 = -5B \rightarrow B = 4/5$$

$$= \int_{-1}^{2} \left[\frac{-4/5}{X-3} + \frac{4/5}{X+2} \right] dx = \left(\frac{-4}{5} \ln|X-3| + \frac{4}{5} \ln|X+2| \right) \Big|_{-1}^{2}$$

$$= \left(\frac{-4}{5} \cancel{\ln 1}^{0} + \frac{4}{5} \ln 4 \right) - \left(\frac{-4}{5} \ln 4 + \frac{4}{5} \cancel{\ln 1}^{0} \right)$$

$$= \frac{8}{5} \ln 4$$

44.) $Y = \dfrac{x^2+2x-1}{x^2-4} = 0 \rightarrow X^2+2X-1 = 0 \rightarrow$

$$X = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2(1)} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm \sqrt{4 \cdot 2}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2} = \sqrt{2} - 1 \quad ;$$

$$\begin{array}{r} 1 \\ X^2-4 \overline{\smash{)}\, X^2+2X-1} \\ -(X^2 -4) \\ \hline 2X+3 \end{array}$$ so

$$\frac{X^2+2X-1}{X^2-4} = 1 + \frac{2X+3}{X^2-4} \; ; \text{ then}$$

$$\text{Area} = \int_{-1}^{\sqrt{2}-1} \frac{x^2+2x-1}{x^2-4} \, dx$$

$$= \int_{-1}^{\sqrt{2}-1} \left[1 + \frac{2X+3}{X^2-4} \right] dx$$

$$= \int_{-1}^{\sqrt{2}-1} \left[1 + \frac{2X+3}{(X-2)(X+2)} \right] dx = \int_{-1}^{\sqrt{2}-1} \left[1 + \frac{A}{X-2} + \frac{B}{X+2} \right] dx$$

$$\rightarrow 2x+3 = A(x+2) + B(x-2) \rightarrow$$

$$x=2: \quad 7 = 4A \rightarrow A = 7/4,$$

$$x=-2: \quad -1 = -4B \rightarrow B = 1/4$$

$$= \int_{-1}^{\sqrt{2}-1} \left[1 + \frac{7/4}{x-2} + \frac{1/4}{x+2} \right] dx$$

$$= \left(x + \frac{7}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| \right) \Big|_{-1}^{\sqrt{2}-1}$$

$$= \left(\sqrt{2} \cancel{-1} + \frac{7}{4}\ln|\sqrt{2}-3| + \frac{1}{4}\ln|\sqrt{2}+1| \right)$$

$$- \left(\cancel{-1} + \frac{7}{4}\ln 3 + \frac{1}{4}\cancelto{0}{\ln 1} \right)$$

$$= \sqrt{2} + \frac{7}{4}\ln|\sqrt{2}-3| + \frac{1}{4}\ln|\sqrt{2}+1| - \frac{7}{4}\ln 3$$

53.) $\int \frac{1}{y(1000-y)}\,dy = \int k\,dt \rightarrow \int \left[\frac{A}{Y} + \frac{B}{1000-Y} \right] dy = kt + C$

$$A(1000-Y) + BY = 1 \quad :$$

$$Y=1000: \quad 1000B = 1 \rightarrow B = \frac{1}{1000},$$

$$Y=0: \quad 1000A = 1 \rightarrow A = \frac{1}{1000}$$

$$\rightarrow \int \left[\frac{1/1000}{Y} + \frac{\frac{1}{1000}}{1000-Y} \right] dy = kt + C$$

$$\rightarrow \frac{1}{1000}\ln|Y| - \frac{1}{1000}\ln|1000-Y| = kt + C$$

$$\rightarrow \frac{1}{1000}\ln Y - \frac{1}{1000}\ln(1000-Y) = kt + C$$

$$\rightarrow \ln Y - \ln(1000-Y) = 1000kt + 1000C$$

$$\rightarrow \ln\left(\frac{Y}{1000-Y}\right) = 1000kt + 1000C$$

$\rightarrow \dfrac{Y}{1000-Y} = e^{1000kt + 1000C} = e^{1000kt} \cancel{e^{1000C}}^{\,C}$

$\rightarrow \boxed{\dfrac{Y}{1000-Y} = Ce^{1000kt}}$; let

$t=0,\ Y=100 \rightarrow \dfrac{100}{1000-100} = Ce^{0} = C(1) = C$

$\rightarrow \underline{C = \dfrac{1\cancel{00}}{9\cancel{00}} = 1/9} \rightarrow$

$\dfrac{Y}{1000-Y} = \dfrac{1}{9}e^{1000kt}$; let

$t=2,\ Y=134 \rightarrow \dfrac{134}{1000-134} = \dfrac{1}{9}e^{2000k} \rightarrow$

$\dfrac{1206}{866} = e^{2000k} \rightarrow \dfrac{603}{433} = e^{2000k} \rightarrow$

$\ln\left(\dfrac{603}{433}\right) = 2000k \rightarrow \underline{k = \dfrac{1}{2000}\ln\left(\dfrac{603}{433}\right)} \rightarrow$

$\boxed{\dfrac{Y}{1000-Y} = \dfrac{1}{9}e^{\frac{1}{2}\ln\left(\frac{603}{433}\right)t}}$; solve for Y:

$9e^{-\frac{1}{2}\ln\left(\frac{603}{433}\right)t}\cdot Y = 1000 - Y \rightarrow$

$9e^{-\frac{1}{2}\ln\left(\frac{603}{433}\right)t}\cdot Y + Y = 1000 \rightarrow$

$\left(9e^{-\frac{1}{2}\ln\left(\frac{603}{433}\right)t} + 1\right)Y = 1000 \rightarrow$

$$Y = \frac{1000}{9e^{-\frac{1}{2}\ln\left(\frac{603}{433}\right)t} + 1}$$

Handout 11

1.) a.) $\int 3\tan 7x \, dx = 3 \cdot \frac{1}{7} \ln|\sec 7x| + C$

b.) $\int \tan^2 x \, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

c.) $\int (1+\sec 5x)^2 dx = \int (1 + 2\sec 5x + \sec^2 5x)\, dx$

$= x + 2 \cdot \frac{1}{5} \ln|\sec 5x + \tan 5x| + \frac{1}{5}\tan 5x + C$

d.) $\int (\tan x + \sec x)^2 dx = \int (\tan^2 x + 2\sec x \tan x + \sec^2 x)\, dx$

$= \int (\sec^2 x - 1 + 2\sec x \tan x + \sec^2 x)\, dx$

$= \int (2\sec^2 x - 1 + 2\sec x \tan x)\, dx$

$= 2\tan x - x + 2\sec x + C$

e.) $\int (\cot x + \cot^2 x)\, dx = \int (\cot x + \csc^2 x - 1)\, dx$

$= \ln|\sin x| - \cot x - x + C$

f.) $\int \csc 3x \cot 3x \, dx = \frac{-1}{3} \csc 3x + C$

g.) $\int \csc 2x \, dx = \frac{1}{2} \ln|\csc 2x - \cot 2x| + C$

h.) $\int \frac{\sec^2 x}{\tan^2 x - 1}\, dx$ (Let $u = \tan x \rightarrow du = \sec^2 x \, dx$)

$= \int \frac{1}{u^2-1}\, du = \int \left(\frac{A}{u+1} + \frac{B}{u-1}\right) du = \cdots = \int \left(\frac{-\frac{1}{2}}{u+1} + \frac{\frac{1}{2}}{u-1}\right) du$

$= \frac{-1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1| + C = \frac{-1}{2}\ln|\tan x + 1| + \frac{1}{2}\ln|\tan x - 1| + C$

2.) $AVE = \frac{1}{\frac{2}{3}\pi - 0} \int_0^{\frac{2}{3}\pi} \tan\left(\frac{x}{2}\right) dx = \frac{3}{2\pi} \cdot 2\ln\left|\sec\left(\frac{x}{2}\right)\right| \Big|_0^{\frac{2}{3}\pi}$

$= \frac{3}{\pi} \ln\left|\sec\frac{\pi}{3}\right| - \cancel{\frac{3}{\pi} \ln|\overset{1}{\sec 0}|} = \frac{3}{\pi}\ln 2$

3.) $Vol = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x \, dx$

$= \pi \tan x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \pi \overset{1}{\tan\left(\frac{\pi}{4}\right)} - \pi \overset{-1}{\tan\left(-\frac{\pi}{4}\right)} = 2\pi$