Math 16B
Sections 4.1, 4.2

Rules for Exponents and Exponential Functions

The Number e

THE NUMBER e (attributed to Leonhard Euler, Swiss Mathematician (1707-1783))

EXAMPLE: Evaluate $(1+1/k)^k$ for the following values of k.

k	$(1+1/k)^k$
1	$2.00000\cdots$
10	$2.59374\cdots$
100	$2.70841\cdots$
1000	$2.71692\cdots$
10,000	$2.71814\cdots$
100,000	$2.71826\cdots$
1,000,000	$2.71828\cdots$
10,000,000	$2.71828\cdots$

CONCLUSION: $\lim_{k\to\infty}(1+1/k)^k = e \approx 2.71828.$

Rules for Exponents

RECALL: The Rules for Exponents

1.) $x^m \cdot x^n = x^{m+n}$

2.) $(x^m)^n = x^{mn}$

3.) $(xy)^m = x^m y^m$

4.) $\frac{x^m}{x^n} = x^{m-n}$

5.) $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$

6.) $x^{-m} = 1/x^m$

<u>Example</u>: Use the Rules of Exponents to simplify each expression.

1.) $\frac{7^{11}}{7^9} = 7^{11-9} = 7^2 = 49$

2.) $3^{-\frac{1}{2}} 3^{\frac{7}{2}} = 3^{-\frac{1}{2}+\frac{7}{2}} = 3^{\frac{6}{2}} = 3^3 = 27$

3.) $(x^{-3})^4 = x^{(-3)(4)} = x^{-12} = \frac{1}{x^{12}}$

4.) $$\frac{(e^3)^5}{e^2e^4} = \frac{e^{15}}{e^6} = e^9$$

5.) $$\left(\frac{x^{-1}}{x^{-5}}\right)^{1/2} = \left(x^{-1-(-5)}\right)^{1/2}$$

$$= \left(x^4\right)^{1/2} = x^2$$

6.) $$\left(\frac{x^{1/3}x^{1/2}}{x}\right)^{12} = \left(\frac{x^{\frac{2}{6}+\frac{3}{6}}}{x^1}\right)^{12}$$

$$= \left(\frac{x^{5/6}}{x^{6/6}}\right)^{12} = \left(x^{-1/6}\right)^{12} = x^{-2} = \frac{1}{x^2}$$

7.) $$\left(\frac{(e^{-3})^2}{e^2\,e^{-4}}\right)^{-5} = \left(\frac{e^{-6}}{e^{-2}}\right)^{-5}$$

$$= \left(e^{-6-(-2)}\right)^{-5} = \left(e^{-4}\right)^{-5} = e^{20}$$

Avoid These Common Mistakes

1.) $(a+b)^m = a^m + b^m$

2.) $a^m a^n = a^{mn}$

3.) $\dfrac{a^m}{a^n} = a^{m/n}$

4.) $\dfrac{1}{a^{-m}+b^{-n}} = a^m + b^n$

5.) $\dfrac{a}{b+c} = \dfrac{a}{b} + \dfrac{a}{c}$

6.) $a^m + a^n = a^{m+n}$

FACT: If $a^m = a^n$, then $m = n$.

Example: Solve for t.

1.) $e^{3t+7} = e^2 e^t \longrightarrow$

$e^{3t+7} = e^{2+t} \longrightarrow 3t+7 = 2+t$

$\longrightarrow 2t = -5 \longrightarrow t = -5/2$

2.) $64^t = 2 \cdot 4^{t+1} \longrightarrow$

(Fake Math: $2 \cdot 4^{t+1} = 8^{t+1}$!!!)

$(2^6)^t = 2 \cdot (2^2)^{t+1} \longrightarrow$

$2^{6t} = 2^1 \cdot 2^{2t+2} \longrightarrow$

$2^{6t} = 2^{2t+3} \longrightarrow 6t = 2t+3 \longrightarrow$

$4t = 3 \longrightarrow t = 3/4$

3.) $9^{t+1} = \left(\frac{1}{3}\right)^{2-t} \longrightarrow$

$$(3^2)^{t+1} = (3^{-1})^{2-t} \longrightarrow$$

$$3^{2t+2} = 3^{-2+t} \longrightarrow$$

$$2t+2 = -2+t \rightarrow \quad t = -4$$

4.) $$\left(\frac{1}{125}\right)^{t-2} = 5 \cdot 5^{2t^2+3} \longrightarrow$$

$$(5^{-3})^{t-2} = 5^1 5^{2t^2+3} \longrightarrow$$

$$5^{-3t+6} = 5^{2t^2+4} \longrightarrow$$

$$-3t+6 = 2t^2+4 \longrightarrow$$

$$0 = 2t^2+3t-2 \longrightarrow$$

$$0 = (2t-1)(t+2) \longrightarrow$$

$$t = \frac{1}{2}, \quad t = -2$$

(Optional Practice Problems)

5.) $5^3 5^{t-1} = \dfrac{5^{2t+1}}{5^4}$

ANS : $t = 5$

6.) $16^{-t} = 4 \cdot 8^{1+t}$

ANS : $t = -5/7$

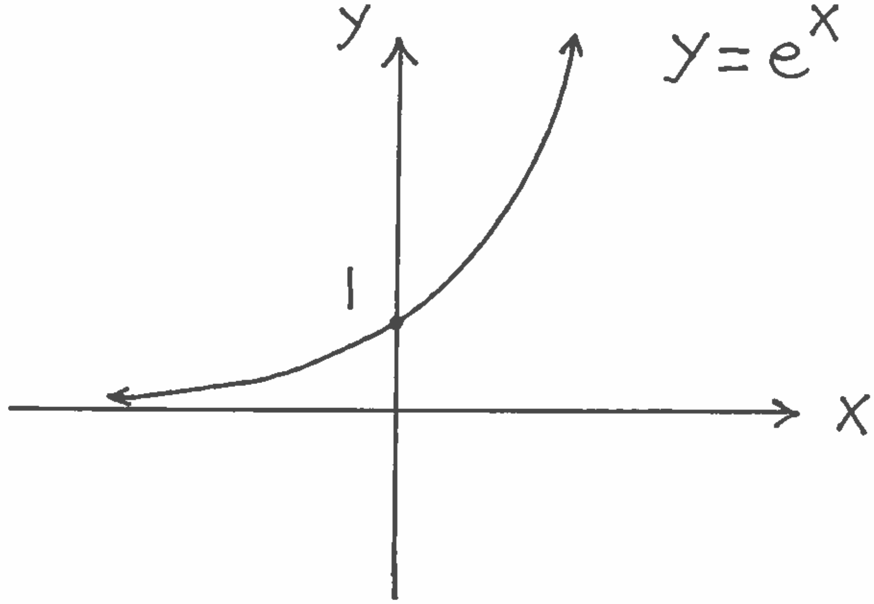
7.) $\left(\frac{1}{27}\right)^{t-2} = 3 \cdot 9^{t+2}$

ANS : $t = 1/5$

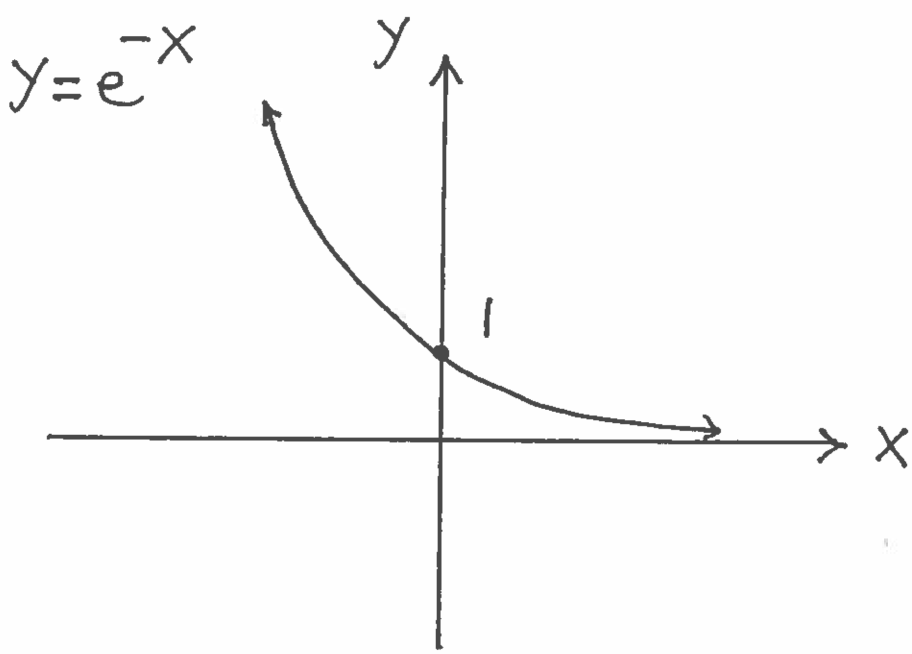
8.) $16^{1-t} = (8^t)^t$

ANS : $t = \frac{2}{3}$, $t = -2$

Graphs of $y = e^x$ and $y = e^{-x}$



NOTE: $e^x > 0$ for all x-values.



NOTE: $e^{-x} > 0$ for all x-values.

Discrete and Continuous Compounding of Interest

Example A: Suppose that \$600 is put in a savings account earning 24% annual interest, which is compounded monthly ($\frac{1}{12}$ of 24% = 2% monthly rate). How much is in the account after 3 months?

# months	amount of \$
1	$600 + 0.02(600) = \$612$
2	$612 + 0.02(612) = \$624.24$
3	$624.24 + 0.02(624.24) \approx \$636.72$

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Discrete and Continuous Compounding of Interest

EXAMPLE : Initially P dollars is deposited into a savings account where interest i is compounded for each of m consecutive interest earning periods. Assume that interest earned remains in the account and that no withdrawals are made and no additional money is deposited in the account. What is the amount A of money in the account after m periods ?

Period	Amount of money
1	$P + i \cdot P = P(1+i)$
2	$P(1+i) + i \cdot P(1+i) = (1+i) \cdot P(1+i) = P(1+i)^2$
3	$P(1+i)^2 + i \cdot P(1+i)^2 = (1+i) \cdot P(1+i)^2 = P(1+i)^3$
4	$P(1+i)^3 + i \cdot P(1+i)^3 = (1+i) \cdot P(1+i)^3 = P(1+i)^4$
$\vdots$	$\vdots$
m	$P(1+i)^m$

So after m periods the total amount of money in the account is $A = P(1+i)^m$. If the annual interest rate is r and interest is compounded n times per year for t years, then $i = r/n$, $m = nt$, and

$$\boxed{A = P(1+r/n)^{nt} \quad \text{(Discrete Interest Formula)}}$$

<u>Example</u>: How much is in the account in Example A after 3 years?

$P = \$600$, $r = 0.24$, $n = 12$, $t = 3 \rightarrow$

$$A = 600\left(1 + \frac{0.24}{12}\right)^{(12)(3)}$$

$$\approx \$1223.93$$

Continuous Compounding of Interest

Begin with $A = P\left(1+\frac{r}{n}\right)^{nt}$ and now let $n \to \infty$.

(RECALL: $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k} = e$)

$$A = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$$

$$= \lim_{n\to\infty} P\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{\left(\frac{n}{r}\right)rt}$$

$$= P\left[\lim_{\frac{n}{r}\to\infty}\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{\left(\frac{n}{r}\right)}\right]^{rt}$$

$$= P\,[e]^{rt}, \text{ i.e.,}$$

$$\boxed{A = Pe^{rt} \quad \text{(Continuous Interest Formula)}}$$

<u>Example</u>: You deposit \$2000 in a savings account earning 7.5% annual interest for 10 years. How much will be in the account if interest is compounded

a.) annually?

b.) monthly?

c.) weekly?

d.) daily?

e.) continuously?

$P = \$2000, \; r = 7.5\% = 0.075, \; t = 10$

a.) annually, so $n = 1$:

$$A = 2000\left(1 + \frac{0.075}{1}\right)^{(1)(10)}$$

$$\approx \$4122.06$$

b.) monthly, so $n = 12$:

$$A = 2000\left(1 + \frac{0.075}{12}\right)^{(12)(10)}$$

$$\approx \$4224.13$$

c.) weekly, so $n = 52$:

$$A = 2000\left(1 + \frac{0.075}{52}\right)^{(52)(10)}$$

$$\approx \$4231.71$$

d.) daily, so $n = 365$:

$$A = 2000\left(1 + \frac{0.075}{365}\right)^{(365)(10)}$$

$$\approx \$4233.67$$

e.) $A = 2000\, e^{(0.075)(10)}$

$$\approx \$4234.00$$

Example: a deposit of \$800 in a savings account grows to \$1800 in 8 years. If interest is compounded monthly, what is the annual interest rate r?

$$A = P\left(1+\frac{r}{n}\right)^{nt} \longrightarrow$$

$$1800 = 800\left(1+\frac{r}{12}\right)^{(12)(8)} \longrightarrow$$

$$\frac{18\cancel{0}\cancel{0}}{8\cancel{0}\cancel{0}} = \left(1+\frac{r}{12}\right)^{96} \longrightarrow$$

$$\frac{9}{4} = \left(1+\frac{r}{12}\right)^{96} \longrightarrow$$

$$\left(\frac{9}{4}\right)^{1/96} = 1+\frac{r}{12} \longrightarrow$$

$$\frac{r}{12} = \left(\frac{9}{4}\right)^{1/96} - 1 \longrightarrow$$

$$r = 12\left[\left(\frac{9}{4}\right)^{1/96} - 1\right] \longrightarrow$$

$$r \approx 0.1018 = 10.18\%$$

Example: An account earning an annual interest rate of 3.5% grows to \$5000 in 12 years. If interest is compounded continuously, what was the initial amount?

$$A = Pe^{rt} \longrightarrow$$

$$5000 = Pe^{(0.035)(12)} \longrightarrow$$

$$5000 = Pe^{0.42} \longrightarrow$$

$$P = \frac{5000}{e^{0.42}} \longrightarrow$$

$$P \approx \$3285.23$$