Math 16B
Kouba
Estimating Definite Integrals

I. TRAPEZOIDAL RULE

1. Divide the interval [a, b] into n equal parts, each of length $\frac{b-a}{n}$.

2. Let $x_0, x_1, x_2, \ldots, x_n$ be the endpoints of the subintervals.

3. An estimate for $\int_a^b f(x)\,dx$ is

$$T_n = \frac{b-a}{2n} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$$

4. Absolute error is $|E_n| \le \frac{(b-a)^3}{12n^2} \cdot \left\{ \max_{a \le x \le b} |f''(x)| \right\}$.

II. SIMPSON'S RULE

1. Divide the interval [a, b] into n equal parts (n MUST be even !) each of length $\frac{b-a}{n}$.

2. Let $x_0, x_1, x_2, \ldots, x_n$ be the endpoints of the subintervals.

3. An estimate for $\int_a^b f(x)\,dx$ is

$$S_n = \frac{b-a}{3n} \Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \Big] .$$

4. Absolute error is $|E_n| \le \frac{(b-a)^5}{180n^4} \cdot \left\{ \max_{a \le x \le b} |f^{(4)}(x)| \right\}$.

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Simpson's Rule

Example:
Use S_4, Simpson's Rule with $n=4$, to estimate the exact value of $\int_{-5}^{-4} \frac{x+1}{x+3}\,dx$.

$f(x) = \frac{x+1}{x+3}$, $n=4$

$-5 \quad \frac{-19}{4} \quad -\frac{9}{2} \quad \frac{-17}{4} \quad -4$

$$S_4 = \frac{-4-(-5)}{3(4)}\left[f(-5) + 4f\left(\frac{-19}{4}\right) + 2f\left(\frac{-9}{2}\right) + 4f\left(\frac{-17}{4}\right) + f(-4)\right]$$

$$= \frac{1}{12}\left[2 + 4\left(\frac{15}{7}\right) + 2\left(\frac{7}{3}\right) + 4\left(\frac{13}{5}\right) + 3\right] \approx 2.3865 ;$$

exact value: $\int_{-5}^{-4} \frac{x+1}{x+3}\,dx = 1 + \ln 4 \approx 2.3863$;

absolute error $|E_4| = \left|\int_{-5}^{-4} \frac{x+1}{x+3}\,dx - S_4\right| = 0.0002$

Question: What should n be in order that S_n, Simpson's Rule with n subdivisions, estimate the exact value of $\int_{-5}^{-4} \frac{x+1}{x+3}\,dx$ with absolute error at most 0.00001 ?

absolute error $|E_n| \le \frac{(b-a)^5}{180\,n^4} \cdot \max_{a \le x \le b} |f^{(4)}(x)| \rightarrow$

$f(x) = \frac{x+1}{x+3}$, $f'(x) = 2(x+3)^{-2}$, $f''(x) = -4(x+3)^{-3}$, $f'''(x) = 12(x+3)^{-4}$, $f^{(4)}(x) = \frac{-48}{(x+3)^5}$ so

$\max_{-5 \le x \le -4} |f^{(4)}(x)| = \frac{48}{|(-4)+3|^5} = 48$; then

$|E_n| \le \frac{(-4-(-5))^5}{180\,n^4} \cdot 48 = \frac{12}{45n^4} \le 0.00001 \rightarrow$

$n^4 \ge \frac{12}{45(0.00001)} \rightarrow n \ge \left[\frac{12}{45(0.00001)}\right]^{1/4} \approx 12.7 \rightarrow$ so use $n=14$!!