Section 7.6

18.) $f(x,y,z) = x^2y^2z^2$ and $x^2+y^2+z^2-1=0$, then

$$F(x,y,z,\lambda) = x^2y^2z^2 - \lambda(x^2+y^2+z^2-1)$$
$$= x^2y^2z^2 - \lambda x^2 - \lambda y^2 - \lambda z^2 + \lambda \rightarrow$$

$F_x = 2xy^2z^2 - 2\lambda x = 2x(y^2z^2-\lambda) = 0$

$\rightarrow x=0$ (No) or $\lambda = y^2z^2$,

$F_y = 2x^2yz^2 - 2\lambda y = 2y(x^2z^2-\lambda) = 0$

$\rightarrow y=0$ (No) or $\lambda = x^2z^2$,

$F_z = 2x^2y^2z - 2\lambda z = 2z(x^2y^2-\lambda) = 0$

$\rightarrow z=0$ (No) or $\lambda = x^2y^2$;

$F_\lambda = -x^2-y^2-z^2+1=0$;

$y^2z^2 = x^2z^2 \rightarrow y^2 = x^2 \rightarrow y=x$

$x^2z^2 = x^2y^2 \rightarrow z^2=y^2 \rightarrow z=y$

$z=y=x$

$\rightarrow -x^2-(x)^2-(x)^2+1=0$

$\rightarrow -3x^2+1=0$

$\rightarrow 3x^2=1 \rightarrow x^2=\frac{1}{3} \rightarrow x = 1/\sqrt{3}$, $y = 1/\sqrt{3}$, $z = 1/\sqrt{3}$, $\lambda = 1/9$, and maximum value

$$f\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{27}.$$

22.) $f(x,y,z) = x^2+y^2+z^2$ and

$x+2z-4=0$, $x+y-8=0$, then

$$F(x,y,z,\lambda,\mu) = x^2+y^2+z^2 - \lambda(x+2z-4) - \mu(x+y-8)$$
$$= x^2+y^2+z^2 - \lambda x - 2\lambda z + 4\lambda - \mu x - \mu y + 8\mu \rightarrow$$

$F_x = 2x - \lambda - \mu = 0 \longrightarrow 2x - (z) - (2y) = 0$

$F_y = 2y - \mu = 0 \rightarrow \mu = 2y$

$F_z = 2z - 2\lambda = 0 \rightarrow \lambda = z$

$F_\lambda = -x - 2z + 4 = 0 \rightarrow$

$F_\mu = -x - y + 8 = 0 \longrightarrow$

$$\begin{cases} 2x - 2y - z = 0 \\ -x \quad\quad - 2z = -4 \\ -x - y \quad\quad = -8 \end{cases}$$

$$\rightarrow \begin{cases} 2x - 2y - z = 0 \\ -2x \quad - 4z = -8 \\ -2x - 2y \quad = -16 \end{cases} \rightarrow \begin{cases} -2y - 5z = -8 \\ -4y - z = -16 \end{cases}$$

$$\rightarrow \begin{cases} 4y + 10z = 16 \\ -4y - z = -16 \end{cases} \rightarrow 9z = 0 \rightarrow z = 0,$$

$y = 4$, $x = 4$, $\lambda = 0$, and minimum value $f(4, 4, 0) = 16 + 16 + 0 = 32$.

24.) $f(x, y, z) = xy + yz$ and

$x + 2y - 6 = 0$, $x - 3z = 0$; then

$$F(x, y, z, \lambda, \mu) = xy + yz - \lambda(x + 2y - 6) - \mu(x - 3z)$$
$$= xy + yz - \lambda x - 2\lambda y + 6\lambda - \mu x + 3\mu z \rightarrow$$

$$\begin{cases} F_x = y - \lambda - \mu = 0 \rightarrow y = \lambda + \mu = \frac{1}{2}x + \frac{1}{2}z - \frac{1}{3}y \\ F_y = x + z - 2\lambda = 0 \rightarrow \lambda = \frac{1}{2}x + \frac{1}{2}z \qquad \frac{4}{3}y = \frac{1}{2}x + \frac{1}{2}z \\ F_z = y + 3\mu = 0 \longrightarrow \mu = -\frac{1}{3}y \qquad \rightarrow y = \frac{3}{8}x + \frac{3}{8}z \\ F_\lambda = -x - 2y + 6 = 0 \\ F_\mu = -x + 3z = 0 \end{cases}$$

$$\begin{Bmatrix} -x - 2\left(\frac{3}{8}x + \frac{3}{8}z\right) + 6 = 0 \\ -x + 3z = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} -\frac{7}{4}x - \frac{3}{4}z + 6 = 0 \\ x = 3z \end{Bmatrix} \quad -\frac{7}{4}(3z) - \frac{3}{4}z = -6 \rightarrow$$

$21z + 3z = 24 \rightarrow z = 1$, $x = 3$, $y = \frac{3}{2}$, $\mu = -\frac{1}{2}$, $\lambda = 2$, and maximum value

$f(3, \frac{3}{2}, 1) = \frac{9}{2} + \frac{3}{2} = 6$.

28.) Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraint: $x + y + z = 120 \rightarrow x + y + z - 120 = 0$;

$$F(x, y, z) = x^2 + y^2 + z^2 - \lambda(x + y + z - 120)$$
$$= x^2 + y^2 + z^2 - \lambda x - \lambda y - \lambda z + 120\lambda$$

$$\begin{cases} F_x = 2x - \lambda = 0 \rightarrow \lambda = 2x \\ F_y = 2y - \lambda = 0 \rightarrow \lambda = 2y \\ F_z = 2z - \lambda = 0 \rightarrow \lambda = 2z \\ F_\lambda = -x - y - z + 120 = 0 \end{cases}$$

$$\left. \begin{array}{l} 2x = 2y \rightarrow x = y \\ 2y = 2z \rightarrow z = y = x \end{array} \right\}$$

$\rightarrow -x - (x) - (x) + 120 = 0 \rightarrow -3x + 120 = 0 \rightarrow$

$x = 40$, $y = 40$, $z = 40$, $\lambda = 80$, and

minimum value $f(40, 40, 40) = 1600 + 1600 + 1600 = 4800$

33.) minimize $f(x,y,z) = (x-2)^2 + (y-1)^2 + (z-1)^2$

Constraint: $x + y + z = 1 \rightarrow x + y + z - 1 = 0$;

$F(x,y,z,\lambda) = (x-2)^2 + (y-1)^2 + (z-1)^2 - \lambda(x + y + z - 1)$

$= (x-2)^2 + (y-1)^2 + (z-1)^2 - \lambda x - \lambda y - \lambda z + \lambda \rightarrow$

$F_x = 2(x-2) - \lambda = 0 \rightarrow \lambda = 2(x-2)$

$F_y = 2(y-1) - \lambda = 0 \rightarrow \lambda = 2(y-1)$

$F_z = 2(z-1) - \lambda = 0 \rightarrow \lambda = 2(z-1)$

$$\left. \begin{array}{l} 2(x-2) = 2(y-1) \\ 2(y-1) = 2(z-1) \end{array} \right\}$$

$F_\lambda = -x - y - z + 1 = 0 \leftarrow \begin{cases} x - 2 = y - 1 \rightarrow y = x - 1 \\ y - 1 = z - 1 \rightarrow z = y = x - 1 \end{cases}$

$\rightarrow -x - (x-1) - (x-1) + 1 = 0$

$\rightarrow -x - x + 1 - x + 1 + 1 = 0 \rightarrow -3x + 3 = 0 \rightarrow x = 1$,

$y = 0$, $z = 0$, $\lambda = -2$, and minimum

value $f(1,0,0) = 1 + 1 + 1 = 3$ so <u>minimum distance</u> is $\sqrt{3}$

Math 16C
Kouba
Worksheet 6

Use Lagrange multipliers to solve each of the following problems.

1.) Minimize $f(x, y, z) = x^2 + y^2 + z^2$

subject to $x - y + z = 0$ and $-x + 2y - z = 3$.

2.) Maximize $f(x, y, z) = 10 - x^2 - 2y^2 - 3z^2$

subject to $x - y = 5$ and $x + y - z = 2$.

3.) The temperature T in degrees Fahrenheit at a point (x, y) on a metal plate is given by

$$T = x^2 - 6x + 9 + y^2 \ .$$

An ant, walking on the plate, traverses a circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant ?

Worksheet 6

1.) Let $F(x,y,z,\lambda,\mu) = x^2+y^2+z^2-\lambda(x-y+z)-\mu(3+x-2y+z)$ then

$$F_x = 2x-\lambda-\mu=0 \qquad 2x+2y+\mu=0 \rightarrow \mu=-2x-2y$$
$$F_y = 2y+\lambda+2\mu=0 \qquad 2y+2z+\mu=0 \rightarrow \mu=-2y-2z$$
$$F_z = 2z-\lambda-\mu=0$$
$$F_\lambda = -x+y-z=0 \qquad -2x-\cancel{2y} = -\cancel{2y}-2z \rightarrow$$
$$F_\mu = -3-x+2y-z=0 \qquad x=z$$

$$\left.\begin{array}{l} -x+y-(x)=0 \\ -3-x+2y-(x)=0 \end{array}\right\} \quad \begin{array}{l} y=2x \\ -3-2x+2y=0 \rightarrow -3-y+2y=0 \end{array}$$

$\rightarrow$ $y=3$, $x=3/2$, $z=3/2$ and

$f\left(\frac{3}{2},3,\frac{3}{2}\right) = \frac{9}{4}+9+\frac{9}{4} = \frac{27}{2}$ is the minimum value.

2.) Let $F(x,y,z,\lambda,\mu) = 10-x^2-2y^2-3z^2-\lambda(x-y-5)-\mu(x+y-z-2)$ then

$$F_x = -2x-\lambda-\mu=0 \qquad -2x-4y-2\mu=0 \rightarrow \mu=-x-2y$$
$$F_y = -4y+\lambda-\mu=0$$
$$F_z = -6z+\mu=0 \rightarrow \mu=6z \longrightarrow 6z=-x-2y \quad \text{or}$$

$$\left.\begin{array}{l} F_\lambda = -x+y+5=0 \\ F_\mu = -x-y+z+2=0 \end{array}\right\} \longrightarrow \left\{\begin{array}{l} x+2y+6z=0 \\ -x+y \qquad =-5 \\ -x-y+z=-2 \end{array}\right.$$

$$\left.\begin{array}{l} 3y+6z=-5 \\ y+7z=-2 \end{array}\right\} \quad -15z=1 \rightarrow z=\frac{-1}{15}, \quad y=\frac{-23}{15},$$

$x=\frac{52}{15}$ and $f\left(\frac{52}{15},\frac{-23}{15},\frac{-1}{15}\right) = \frac{-101}{15}$ is the maximum value.

3.) Maximize temperature $T = x^2 - 6x + 9 + y^2$
subject to $x^2 + y^2 = 25$:

Let $F(x, y, \lambda) = (x^2 - 6x + 9 + y^2) - \lambda(x^2 + y^2 - 25)$ then

$F_x = 2x - 6 - 2\lambda x = 0$ } $\lambda = \dfrac{x-3}{x}$

$F_y = 2y - 2\lambda y = 0$ } $2y(1-\lambda) = 0 \rightarrow \underline{\underline{y=0}}$ or $\underline{\underline{\lambda = 1}}$

$F_\lambda = -x^2 - y^2 + 25 = 0$

$\hookrightarrow x^2 + y^2 = 25$;

if $\lambda = 1$ then $1 = \dfrac{x-3}{x} \rightarrow x = x - 3$

$\rightarrow 0 = -3$ (impossible!) ;

if $y = 0$ then $x^2 + (0)^2 = 25 \rightarrow x = \pm 5$;

at $(5,0)$ temperature $T = 4°$ is the lowest

and

at $(-5,0)$ temperature $T = 64°$ is the highest.