Section 10.4 (cont'd.)

26.) $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ so

$$e^{-x} = e^{(-x)} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n!} \; ; \quad \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \lim_{n \to \infty} \frac{|x|}{n+1} = 0 < 1$$

for all $x$ and $R = +\infty$.

28.) $e^{-2x} = e^{(-2x)} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots$

$$= 1 - 2x + \frac{2^2}{2!}x^2 - \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!} x^n \; ; \quad \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n \to \infty} \frac{2^{n+1} \cdot |x|^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n \cdot |x|^n} = \lim_{n \to \infty} \frac{2|x|}{n+1} = 0 < 1$$

for all $x$ and $R = +\infty$.

29.) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ so

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n ;$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} = |x| < 1 \rightarrow -1 < x < 1$$

and $R = 1$ .

32.) $f(x) = x^{1/2}$, $c = 4$, $a_n = \dfrac{f^{(n)}(c)}{n!}$ ,

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$f''(x) = \frac{-1}{2^2} x^{-3/2}$$

$$f'''(x) = \frac{1\cdot 3}{2^3} x^{-5/2}$$

$$f^{(4)}(x) = \frac{-1\cdot 3\cdot 5}{2^4} x^{-7/2}$$

$$f^{(5)}(x) = \frac{1\cdot 3\cdot 5\cdot 7}{2^5} x^{-9/2}$$

$$\vdots$$

$$a_0 = f(4) = 2$$

$$a_1 = f'(4) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^2}$$

$$a_2 = \frac{f''(4)}{2!} = \frac{-1}{2^2} \cdot \frac{1}{2^3} \cdot \frac{1}{2!} = \frac{-1}{2^5 \cdot 2!}$$

$$a_3 = \frac{f'''(4)}{3!} = \frac{1\cdot 3}{2^3} \cdot \frac{1}{2^5} \cdot \frac{1}{3!} = \frac{1\cdot 3}{2^8 \cdot 3!}$$

$$a_4 = \frac{f^{(4)}(4)}{4!} = \frac{-1\cdot 3\cdot 5}{2^4} \cdot \frac{1}{2^7} \cdot \frac{1}{4!} = \frac{-1\cdot 3\cdot 5}{2^{11} \cdot 4!}$$

$$a_5 = \frac{f^{(5)}(4)}{5!} = \frac{1\cdot 3\cdot 5\cdot 7}{2^5} \cdot \frac{1}{2^9} \cdot \frac{1}{5!} = \frac{1\cdot 3\cdot 5\cdot 7}{2^{14} \cdot 5!}$$

$$\vdots$$

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 + \cdots$$

$$= 2 + \frac{1}{4}(x-4) + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2^{3n-1} \cdot n!} (x-4)^n ;$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots (2(n+1)-3) \cdot |x-4|^{n+1}}{2^{3(n+1)-1} \cdot (n+1)!} \cdot \frac{2^{3n-1} \cdot n!}{1\cdot 3\cdot 5 \cdots (2n-3) \cdot |x-4|^n}$$

$$= \lim_{n\to\infty} \frac{2n-1}{n+1} \cdot \frac{1}{2^3} \cdot |x-4| = \frac{1}{4}|x-4| < 1 \rightarrow |x-4| < 4 \rightarrow$$

$-4 < x-4 < 4 \rightarrow 0 < x < 8$ and $R = 4$ .

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \; ; \quad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n\to\infty} \frac{(n+2)|x|^{n+1}}{(n+1)\,|x|^n} = |x| < 1 \rightarrow$$

$-1 < x < 1$ and $R = 1$.

34.) $f(x) = (1+x)^{1/2}$, $c = 0$, $a_n = \dfrac{f^{(n)}(c)}{n!}$,

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f''(x) = \frac{-1}{2^2}(1+x)^{-3/2}$

$f'''(x) = \frac{1\cdot 3}{2^3}(1+x)^{-5/2}$

$f^{(4)}(x) = \frac{-1\cdot 3\cdot 5}{2^4}(1+x)^{-7/2}$

$f^{(5)}(x) = \frac{1\cdot 3\cdot 5\cdot 7}{2^5}(1+x)^{-9/2}$

$\vdots$

$a_0 = f(0) = 1$

$a_1 = f'(0) = \frac{1}{2}$

$a_2 = \frac{f''(0)}{2!} = \frac{-1}{2^2\, 2!}$

$a_3 = \frac{f'''(0)}{3!} = \frac{1\cdot 3}{2^3\cdot 3!}$

$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{-1\cdot 3\cdot 5}{2^4\cdot 4!}$

$a_5 = \frac{f^{(5)}(0)}{5!} = \frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}$

$\vdots$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{1\cdot 3}{2^3\cdot 3!}x^3 - \frac{1\cdot 3\cdot 5}{2^4\cdot 4!}x^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}x^5 - \cdots$$

$$= 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n\cdot n!}\, x^n \; ;$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2(n+1)-3)\,|x|^{n+1}}{2^{n+1}\cdot(n+1)!} \cdot \frac{2^n\cdot n!}{1\cdot 3\cdot 5\cdots(2n-3)\cdot |x|^n}$$

$$= \lim_{n\to\infty} \frac{2n-1}{n+1} \cdot \frac{|x|}{2} = 2 \cdot \frac{|x|}{2} = |x| < 1 \rightarrow$$

$-1 < x < 1$ and $R = 1$ .

36.) $f(x) = (1+x)^{1/3}$, $c = 0$, $a_n = \dfrac{f^{(n)}(c)}{n!}$ ,

$f'(x) = \frac{1}{3}(1+x)^{-2/3}$

$f''(x) = \frac{-2}{3^2}(1+x)^{-5/3}$

$f'''(x) = \frac{2\cdot 5}{3^3}(1+x)^{-8/3}$

$f^{(4)}(x) = \frac{-2\cdot 5\cdot 8}{3^4}(1+x)^{-11/3}$

$f^{(5)}(x) = \frac{2\cdot 5\cdot 8\cdot 11}{3^5}(1+x)^{-14/3}$

$\vdots$

$a_0 = f(0) = 1$

$a_1 = f'(0) = \frac{1}{3}$

$a_2 = \frac{f''(0)}{2!} = \frac{-2}{3^2 2!}$

$a_3 = \frac{f'''(0)}{3!} = \frac{2\cdot 5}{3^3 \cdot 3!}$

$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{-2\cdot 5\cdot 8}{3^4 \cdot 4!}$

$a_5 = \frac{f^{(5)}(0)}{5!} = \frac{2\cdot 5\cdot 8\cdot 11}{3^5 \cdot 5!}$

$\vdots$

$$(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{2}{3^2 2!}x^2 + \frac{2\cdot 5}{3^3\cdot 3!}x^3 - \frac{2\cdot 5\cdot 8}{3^4\cdot 4!}x^4 + \cdots$$

$$= 1 + \frac{1}{3}x + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{2\cdot 5\cdot 8\cdots(3n-4)}{3^n \cdot n!} x^n \; ; \quad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n\to\infty} \frac{2\cdot 5\cdot 8\cdots(3(n+1)-4)}{3^{n+1}\cdot(n+1)!} |x|^{n+1} \cdot \frac{3^n \cdot n!}{2\cdot 5\cdot 8\cdots(3n-4)\cdot |x|^n}$$

$$= \lim_{n\to\infty} \frac{3n-1}{n+1} \cdot \frac{|x|}{3} = 3 \cdot \frac{|x|}{3} = |x| < 1 \rightarrow$$

$-1 < x < 1$ and $R = 1$ .

41.) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ so

$f(x) = e^{x^3} = 1 + (x^3) + \frac{(x^3)^2}{2!} + \frac{(x^3)^3}{3!} + \frac{(x^3)^4}{4!} + \cdots$

$= 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots$

$= \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$

43.) $3x^2 e^{x^3} = D\left(e^{x^3}\right)$

$= D\left(1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots\right)$

$= 0 + 3x^2 + \frac{6x^5}{2!} + \frac{9x^8}{3!} + \frac{12x^{11}}{4!} + \cdots$

$= \sum_{n=1}^{\infty} \frac{3n \cdot x^{3n-1}}{n!}$

45.) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ so

$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$

$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$ so

$\frac{1}{1+x^4} = 1 - (x^4) + (x^4)^2 - (x^4)^3 + (x^4)^4 - (x^4)^5 + \cdots$

$= 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots = \sum_{n=0}^{\infty} (-1)^n x^{4n}$

46.) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$ so

$\frac{2x}{1+x} = 2x \cdot \frac{1}{1+x} = 2x\left(1 - x + x^2 - x^3 + x^4 - \cdots\right)$

$= 2x - 2x^2 + 2x^3 - 2x^4 + 2x^5 - 2x^6 + \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \cdot 2x^{n+1}$

48.) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ so

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n}$

47.) (Use 48.)

$\ln(1+x^2) = (x^2) - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \frac{(x^2)^4}{4} + \cdots$

$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$

I.) $f(x) = \ln x$, $c = 1$, $a_n = \dfrac{f^{(n)}(c)}{n!}$,

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2 \cdot 1 \cdot x^{-3}$

$f^{(4)}(x) = -3 \cdot 2 \cdot 1 \cdot x^{-4}$

$f^{(5)}(x) = 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^{-5}$

$\vdots$

$a_0 = f(1) = \ln 1 = 0$

$a_1 = f'(1) = 1$

$a_2 = \frac{f''(1)}{2!} = \frac{-1}{2!} = -\frac{1}{2}$

$a_3 = \frac{f'''(1)}{3!} = \frac{2 \cdot 1}{3!} = \frac{1}{3}$

$a_4 = \frac{f^{(4)}(1)}{4!} = \frac{-3 \cdot 2 \cdot 1}{4!} = \frac{-1}{4}$

$a_5 = \frac{f^{(5)}(1)}{5!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{5!} = \frac{1}{5}$

$\vdots$

$$\ln x = 0 + 1 \cdot (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} \cdot (x-1)^n \; ; \quad \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n \to \infty} \frac{|x-1|^{n+1}}{n+1} \cdot \frac{n}{|x-1|^n} = |x-1| < 1 \rightarrow -1 < x-1 < 1 \rightarrow$$

$0 < x < 2$ and $R = 1$.