Math 16C
Kouba
Worksheet 10

1.) Use any method (repeated differentiation method or short cuts using well know Maclaurin series) to find the first four nonzero terms of the Taylor series centered at c for each of the following functions.

a.) $f(x) = x^2 + 3 + \sin(x^2)$ and $c = 0$

b.) $f(x) = e^{-x} - \cos\sqrt{x}$ and $c = 0$

c.) $f(x) = x^3 \cos 2x$ and $c = 0$

d.) $f(x) = (1 + x + x^2) \cdot \ln(1 + x)$ and $c = 0$

e.) $f(x) = \dfrac{x^5}{1 - 2x^3}$ and $c = 0$

f.) $f(x) = x + \sqrt{x + 4}$ and $c = -3$

g.) $f(x) = \tan(\pi x/4)$ and $c = 1$

2.) Determine the common function represented by (equal to) each of the following Taylor series.

a.) $1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots$

b.) $x^2 + x^3 + x^4/2! + x^5/3! + x^6/4! + \cdots$

c.) $1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$

d.) $x^4 - x^6 + x^8 - x^{10} + x^{12} - \cdots$

Worksheet 10

1.) a.) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ so that

$$f(x) = x^2 + 3 + \sin(x^2)$$

$$= x^2 + 3 + \left[(x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots\right]$$

$$= x^2 + 3 + x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

$$= 3 + 2x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

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b.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ so that

$$f(x) = e^{-x} - \cos\sqrt{x}$$

$$= \left[1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \cdots\right]$$

$$- \left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \frac{(\sqrt{x})^8}{8!} - \cdots\right]$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots$$

$$-1 + \frac{x}{2} - \frac{x^2}{24} + \frac{x^3}{720} - \frac{x^4}{40,320} + \cdots$$

$$= \frac{-1}{2}x + \frac{11}{24}x^2 - \frac{119}{720}x^3 + \frac{1679}{40,320}x^4 + \cdots$$

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c.) $f(x) = x^3 \cos(2x)$

$$= x^3 \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right]$$

$$= x^3\left[1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right]$$

$$= x^3 - 2x^5 + \frac{2}{3}x^7 - \frac{4}{45}x^9 + \cdots$$

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d.) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ so

$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3$ and (integrate)

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ then

$$f(x) = (1+x+x^2)\ln(1+x)$$

$$= (1+x+x^2)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) \quad \text{(expand)}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$+ x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots$$

$$+ x^3 - \frac{x^4}{2} + \cdots$$

$$= x + \frac{1}{2}x^2 + \frac{5}{6}x^3 - \frac{5}{12}x^4 + \cdots$$

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e.) $f(x) = \frac{x^5}{1-2x^3} = x^5 \cdot \frac{1}{1-(2x^3)}$

$$= x^5\left[1 + (2x^3) + (2x^3)^2 + (2x^3)^3 + \cdots\right]$$

$$= x^5\left[1 + 2x^3 + 4x^6 + 8x^9 + \cdots\right]$$

$$= x^5 + 2x^8 + 4x^{11} + 8x^{14} + \cdots$$

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f.) $f(x) = x + (x+4)^{1/2} \rightarrow f'(x) = 1 + \frac{1}{2}(x+4)^{-1/2}$,

$f''(x) = \frac{-1}{4}(x+4)^{-3/2}$, $f'''(x) = \frac{3}{8}(x+4)^{-5/2}$, ... ,

$a_n = \frac{f^{(n)}(-3)}{n!}$ so $a_0 = \frac{f(-3)}{0!} = \frac{-2}{1} = -2$,

$a_1 = \frac{f'(-3)}{1!} = \frac{\frac{3}{2}}{1} = \frac{3}{2}$, $a_2 = \frac{f''(-3)}{2!} = \frac{\frac{-1}{4}}{2} = \frac{-1}{8}$,

$a_3 = \frac{f'''(-3)}{3!} = \frac{\frac{3}{8}}{6} = \frac{1}{16}$, ... then

$$f(x) = x + (x+4)^{1/2} = -2 + \frac{3}{2}(x+3) - \frac{1}{8}(x+3)^2 + \frac{1}{16}(x+3)^3 + \cdots$$

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g.) $f(x) = \tan\left(\frac{\pi}{4}x\right) \rightarrow f'(x) = \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}x\right)$,

$f''(x) = \frac{\pi}{2}\sec\left(\frac{\pi}{4}x\right)\cdot\sec\left(\frac{\pi}{4}x\right)\tan\left(\frac{\pi}{4}x\right)\cdot\frac{\pi}{4} = \frac{\pi^2}{8}\sec^2\left(\frac{\pi}{4}x\right)\tan\left(\frac{\pi}{4}x\right)$,

$f'''(x) = \frac{\pi^2}{8}\sec^2\left(\frac{\pi}{4}x\right)\cdot\sec^2\left(\frac{\pi}{4}x\right)\cdot\frac{\pi}{4}$

$+ \frac{\pi^2}{8}\tan\left(\frac{\pi}{4}x\right)\cdot 2\sec\left(\frac{\pi}{4}x\right)\cdot\sec\left(\frac{\pi}{4}x\right)\tan\left(\frac{\pi}{4}x\right)\cdot\frac{\pi}{4}$

$= \frac{\pi^3}{32}\sec^4\left(\frac{\pi}{4}x\right) + \frac{\pi^3}{16}\sec^2\left(\frac{\pi}{4}x\right)\tan^2\left(\frac{\pi}{4}x\right)$, ... ,

$a_n = \frac{f^{(n)}(1)}{n!}$ so $a_0 = \frac{f(1)}{0!} = \frac{\tan\left(\frac{\pi}{4}\right)}{1} = \frac{1}{1} = 1$,

$a_1 = \frac{f'(1)}{1!} = \frac{\frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right)}{1} = \frac{\pi}{4}\cdot 2 = \frac{\pi}{2}$,

$a_2 = \frac{f''(1)}{2!} = \frac{\pi^2}{16}\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}\cdot 2\cdot 1 = \frac{\pi^2}{8}$,

$a_3 = \frac{f'''(1)}{3!} = \frac{1}{6}\left[\frac{\pi^3}{32}\sec^4\left(\frac{\pi}{4}\right) + \frac{\pi^3}{16}\sec^2\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right)\right]$

$$= \frac{1}{6}\left[\frac{\pi^3}{32}\cdot 4 + \frac{\pi^3}{16}\cdot 2\cdot 1\right] = \frac{\pi^3}{24}, \ldots \text{ then}$$

$$f(x) = \tan\left(\frac{\pi}{4}x\right) = 1 + \frac{\pi}{2}(x-1) + \frac{\pi^2}{8}(x-1)^2 + \frac{\pi^3}{24}\cdot(x-1)^3 + \cdots.$$

2.) a.) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$

b.) $x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!}$

$$= x^2\cdot\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right] = x^2 e^x$$

c.) $1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$

d.) $x^4 - x^6 + x^8 - x^{10} + x^{12} - \cdots$

$$= x^4\cdot\left[1 - x^2 + x^4 - x^6 + x^8 - \cdots\right]$$

$$= x^4\cdot\left[1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots\right]$$

$$= x^4\cdot\frac{1}{1-(-x^2)} = \frac{x^4}{1+x^2}$$