Section 10.5 (cont'd.)

21.) (See problem 10.)

$$\int_0^1 e^{-x^2}dx \approx \int_0^1 \left(1-x^2+\frac{x^4}{2}-\frac{x^6}{6}\right) dx$$

$$= \left(x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}\right)\Big|_0^1$$

$$= 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42} \approx 0.742857142857$$

22.) (See problem 2.)

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} \ln(x^2+1)\,dx \approx \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(x^2-\frac{(x^2)^2}{2}+\frac{(x^2)^3}{3}\right) dx$$

$$= \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(x^2-\frac{x^4}{2}+\frac{x^6}{3}\right) dx$$

$$= \left(\frac{x^3}{3}-\frac{x^5}{10}+\frac{x^7}{21}\right)\Big|_{-1/4}^{1/4}$$

$$= \left(\frac{\left(\frac{1}{4}\right)^3}{3}-\frac{\left(\frac{1}{4}\right)^5}{10}+\frac{\left(\frac{1}{4}\right)^7}{21}\right)-\left(\frac{\left(-1/4\right)^3}{3}-\frac{\left(-1/4\right)^5}{10}+\frac{\left(-1/4\right)^7}{21}\right)$$

$$\approx 0.0102271670 39$$

23.) See p. 687 and formula for $(1+x)^k$ with $k = \frac{-1}{2}$:

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}x^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}x^3$$

$$+ \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{4!}x^4 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{5!}x^5 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}}{6!}x^6 + \cdots$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$ so

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots$$ and

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6$$ so

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1+x^2}}\,dx \approx \int_0^{\frac{1}{2}} \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6\right) dx$$

$$= \left(x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7\right)\Big|_0^{1/2}$$

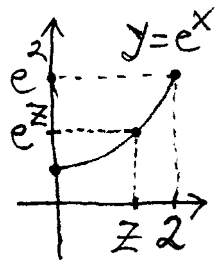
$$= \left(\frac{1}{2}\right) - \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5 - \frac{5}{112}\left(\frac{1}{2}\right)^7$$

$$\approx 0.48116164345$$

25.) $f(x)=e^x$, $c=1$, $[0,2]$ $\rightarrow$

$f'(x)=e^x=f''(x)=f'''(x)=\cdots$ so $f^{(n)}(x)=e^x$ and

$$|f(x)-P_n(x)|=|R_n(x)|=\left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$$



$$=\frac{e^z}{(n+1)!}|x-1|^{n+1}$$ where z is between 1 and x and $0\le x\le 2$ so $0\le z\le 2$

$$\le\frac{e^2}{(n+1)!}(1)^{n+1}$$

$$\le\frac{3^2}{(n+1)!}=\frac{9}{(n+1)!}\le .001$$ so

$n+1=8$ works $\rightarrow$ $\boxed{n=7}$

26.) $f(x)=\frac{1}{x}$, $c=1$, $[1,\tfrac{3}{2}]$ $\rightarrow$

$f'(x)=\frac{-1}{x^2}$, $f''(x)=\frac{+2}{x^3}$, $f'''(x)=\frac{-3\cdot 2}{x^4}$, $f^{(4)}(x)=\frac{+4\cdot 3\cdot 2}{x^5}$,

$\ldots$, $f^{(n)}(x)=\frac{(-1)^n\, n!}{x^{n+1}}$ and

$$|f(x)-P_n(x)|=|R_n(x)|=\left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$$

$$=\left|\frac{\frac{(-1)^{n+1}(n+1)!}{z^{n+2}}}{(n+1)!}(x-1)^{n+1}\right|$$

$$=\frac{\cancel{(n+1)!}}{|z|^{n+2}}\cdot\frac{1}{\cancel{(n+1)!}}|x-1|^{n+1}$$ (where z is between x and $c=1$ and $1\le x\le \tfrac{3}{2}$)

$$= \frac{|x-1|^{n+1}}{|z|^{n+2}} \leq \frac{\left|\frac{3}{2}-1\right|^{n+1}}{(1)^{n+2}}$$

$$= \left(\frac{1}{2}\right)^{n+1} \leq 0.001 \rightarrow \ln\left(\frac{1}{2}\right)^{n+1} \leq \ln(0.001)$$

$$\rightarrow (n+1)\ln(1/2) \leq \ln(0.001)$$

$$\rightarrow n+1 \geq \frac{\ln(0.001)}{\ln(1/2)} \quad (\text{since } \ln(1/2) < 0)$$

$$\rightarrow n+1 \geq 9.965784 \rightarrow n \geq 8.965784$$

$$\rightarrow \boxed{n \geq 9} \text{ works.}$$

27.) $f(x) = e^{-x}$, $c = 0$, $[0,1] \rightarrow$

$f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x}$, $\ldots$,

$f^{(n)}(x) = \pm e^{-x}$ so

$$|f(x) - P_5(x)| = |R_5(x)| = \left|\frac{f^{(5+1)}(z)}{(5+1)!}(x-c)^{5+1}\right|$$

$$= \frac{e^{-z}}{(5+1)!}|x-0|^{5+1} \quad \text{where } 0 \leq z \leq x \text{ and } 0 \leq x \leq 1$$

$$\leq \frac{e^{-0}}{6!}(1)^6$$

$$= \frac{1}{6!}$$

$$= \boxed{.001388888}.$$

28.) $f(x) = \frac{1}{x}$, $c = 1$, $[1, 3/2] \longrightarrow$

$f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = \frac{-3 \cdot 2}{x^4}$, $\cdots$,

$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$ then

$$|f(x) - P_5(x)| = |R_5(x)| = \left| \frac{f^{(5+1)}(z)}{(5+1)!} (x-1)^{5+1} \right|$$

$$= \frac{\frac{6!}{|z|^7}}{6!} \cdot |x-1|^6$$ where $1 \le z \le x$ and $1 \le x \le 3/2$

$$= \frac{|x-1|^6}{|z|^7}$$

$$\le \frac{\left|\frac{3}{2}-1\right|^6}{(1)^7}$$

$$= \left(\frac{1}{2}\right)^6$$

$$= \boxed{.015625}$$

Math 16C
Kouba
Worksheet 11

1.) List the first 5 terms (starting with $n = 1$) of each sequence. Determine the limit of each sequence or state that the limit does not exist.

a.) $a_n = \cos(n\pi)$

b.) $a_n = \dfrac{\cos(n\pi)}{n}$

c.) $a_n = n \cdot \cos(n\pi)$

d.) $a_n = \sin(n\pi)$

e.) $a_n = \sin(\pi/n)$

f.) $a_n = n \cdot \sin(1/n)$

g.) $a_n = \tan((\pi/4) + n(\pi/2))$

2.) Determine if each series converges or diverges. Briefly explain and name the test that you are using.

a.) $\displaystyle\sum_{n=1}^{\infty} \cos(n\pi)$

b.) $\displaystyle\sum_{n=1}^{\infty} \sin((\pi/2) + (1/n))$

c.) $\displaystyle\sum_{n=1}^{\infty} \frac{\tan((4n+1)(\pi/4))}{n^3}$

d.) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

e.) $\displaystyle\sum_{n=1}^{\infty} \frac{1-\cos(n\pi)}{n^2}$

3.) Determine the degree n of the Taylor polynomial $p_n(x)$ centered at $c = 0$ that will estimate the value of the function $f(x) = \ln(1-x)$ with absolute error at most 0.001 on the interval $[-1/2, 0]$.

Worksheet 11

(1.) (a.) $a_n = \cos(n\pi) \longrightarrow$

$$\left.\begin{aligned} a_1 &= \cos\pi = -1 \\ a_2 &= \cos 2\pi = +1 \\ a_3 &= \cos 3\pi = -1 \\ a_4 &= \cos 4\pi = +1 \\ a_5 &= \cos 5\pi = -1 \end{aligned}\right\} \quad \lim_{n\to\infty} \cos(n\pi)$$

does not exist

(b.) $a_n = \cos(n\pi) \div n$

$$\left.\begin{aligned} a_1 &= \cos\pi \div 1 = -1 \\ a_2 &= \cos 2\pi \div 2 = +1/2 \\ a_3 &= \cos 3\pi \div 3 = -1/3 \\ a_4 &= \cos 4\pi \div 4 = +1/4 \\ a_5 &= \cos 5\pi \div 5 = -1/5 \end{aligned}\right\} \quad \lim_{n\to\infty} \frac{\cos(n\pi)}{n} = 0$$

(c.) $a_n = n\cos(n\pi) \longrightarrow$

$$\left.\begin{aligned} a_1 &= 1\cdot\cos\pi = -1 \\ a_2 &= 2\cos 2\pi = +2 \\ a_3 &= 3\cos 3\pi = -3 \\ a_4 &= 4\cos 4\pi = +4 \\ a_5 &= 5\cos 5\pi = -5 \end{aligned}\right\} \quad \lim_{n\to\infty} n\cos(n\pi)$$

does not exist

(d.) $a_n = \sin(n\pi) \longrightarrow$

$$\left.\begin{aligned} a_1 &= \sin\pi = 0 \\ a_2 &= \sin 2\pi = 0 \\ a_3 &= \sin 3\pi = 0 \\ a_4 &= \sin 4\pi = 0 \\ a_5 &= \sin 5\pi = 0 \end{aligned}\right\} \quad \lim_{n\to\infty} \sin(n\pi) = 0$$

(e.) $a_n = \sin(\pi/n)$

$a_1 = \sin \pi = 0$

$a_2 = \sin(\pi/2) = 1$

$a_3 = \sin(\pi/3) = \sqrt{3}/2 \approx .866025403$

$a_4 = \sin(\pi/4) = \sqrt{2}/2 \approx .707106781$

$a_5 = \sin(\pi/5) = .587785252$

$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin(0) = 0$

(f.) $a_n = n \sin\left(\frac{1}{n}\right)$

$a_1 = 1 \cdot \sin 1 = .8414709$

$a_2 = 2 \cdot \sin\left(\frac{1}{2}\right) = .958851$

$a_3 = 3 \sin\left(\frac{1}{3}\right) = .981584$

$a_4 = 4 \sin\left(\frac{1}{4}\right) = .9896158$

$a_5 = 5 \sin\left(\frac{1}{5}\right) = .9933466$

$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1$

(Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$)

(g.) $a_n = \tan\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$

$a_1 = \tan\left(\frac{3\pi}{4}\right) = -1$

$a_2 = \tan\left(\frac{5\pi}{4}\right) = +1$

$a_3 = \tan\left(\frac{7\pi}{4}\right) = -1$

$a_4 = \tan\left(\frac{9\pi}{4}\right) = +1$

$a_5 = \tan\left(\frac{11\pi}{4}\right) = -1$

$\lim_{n\to\infty} \tan\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$

does not exist.

2. a. $\lim_{n\to\infty} \cos(n\pi) \neq 0$ (SEE 1. a.)

so $\sum_{n=1}^{\infty} \cos(n\pi)$ diverges by nth term test

b. $\lim_{n\to\infty} \sin\left(\frac{\pi}{2}+\frac{1}{n}\right) = \sin\frac{\pi}{2} = 1 \neq 0$

so $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2}+\frac{1}{n}\right)$ diverges by nth term test

c. $\sum_{n=1}^{\infty} \frac{\tan\left(\frac{(4n+1)\pi}{4}\right)}{n^3}$

$= \frac{\tan\left(\frac{5\pi}{4}\right)}{1^3} + \frac{\tan\left(\frac{9\pi}{4}\right)}{2^3} + \frac{\tan\left(\frac{13\pi}{4}\right)}{3^3} + \frac{\tan\left(\frac{17\pi}{4}\right)}{4^3} + \cdots$

$= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$

converges since this is a

p-series with $p = 3 > 1$.

d. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$; apply ratio test :

$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)\cdot n^n}$

$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

$= e > 1$

so series diverges.

e.) $\displaystyle\sum_{n=1}^{\infty} \frac{1-\cos(n\pi)}{n^2}$

$$= \frac{1-\cos\pi}{1^2} + \frac{1-\cos 2\pi}{2^2} + \frac{1-\cos 3\pi}{3^2} + \frac{1-\cos 4\pi}{4^2} + \cdots$$

$$= \frac{1-(-1)}{1^2} + \frac{1-(1)}{2^2} + \frac{1-(-1)}{3^2} + \frac{1-(-1)}{4^2} + \frac{1-(1)}{5^2} + \cdots$$

$$= \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \cdots = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)$$

$$\boxed{\leq} \; 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is a convergent p-series since $p = 2 > 1$; since $0 \leq \displaystyle\sum_{n=1}^{\infty} \frac{1-\cos(n\pi)}{n^2} \leq 2\sum_{n=1}^{\infty} \frac{1}{n^2}$; thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1-\cos(n\pi)}{n^2}$ converges since its value is smaller.

(3.) $f(x) = \ln(1-x)$, $c = 0$, $\left[-\frac{1}{2}, 0\right]$ →

$$f'(x) = \frac{-1}{1-x} = -(1-x)^{-1}, \quad f''(x) = -(1-x)^{-2},$$

$$f'''(x) = -2(1-x)^{-3}, \quad f^{(4)}(x) = -3\cdot 2\,(1-x)^{-4}, \ldots$$

$$f^{(n)}(x) = -(n-1)!\,(1-x)^{-n} = \frac{-(n-1)!}{(1-x)^{n}}$$ and

$$f^{(n+1)}(x) = -n!\,(1-x)^{-(n+1)} = \frac{-n!}{(1-x)^{n+1}}$$; then

the absolute error is

$$|f(x) - P_n(x)| = |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1} \right|$$

$$= \frac{\dfrac{n!}{|1-z|^{n+1}}}{(n+1)!} \, |x-0|^{n+1}$$

$$= \frac{1}{n+1} \cdot \frac{|x|^{n+1}}{|1-z|^{n+1}} \qquad (x, z \text{ are in } [-\tfrac{1}{2}, 0])$$

$$\leq \frac{1}{n+1} \cdot \frac{\left|\frac{-1}{2}\right|^{n+1}}{|1-0|^{n+1}}$$

$$= \frac{1}{n+1} \; \frac{\left(\frac{1}{2}\right)^{n+1}}{(1)^{n+1}}$$

$$= \frac{1}{n+1} \left(\frac{1}{2}\right)^{n+1} \leq 0.001$$;

by calculator $\boxed{n=7}$ works .