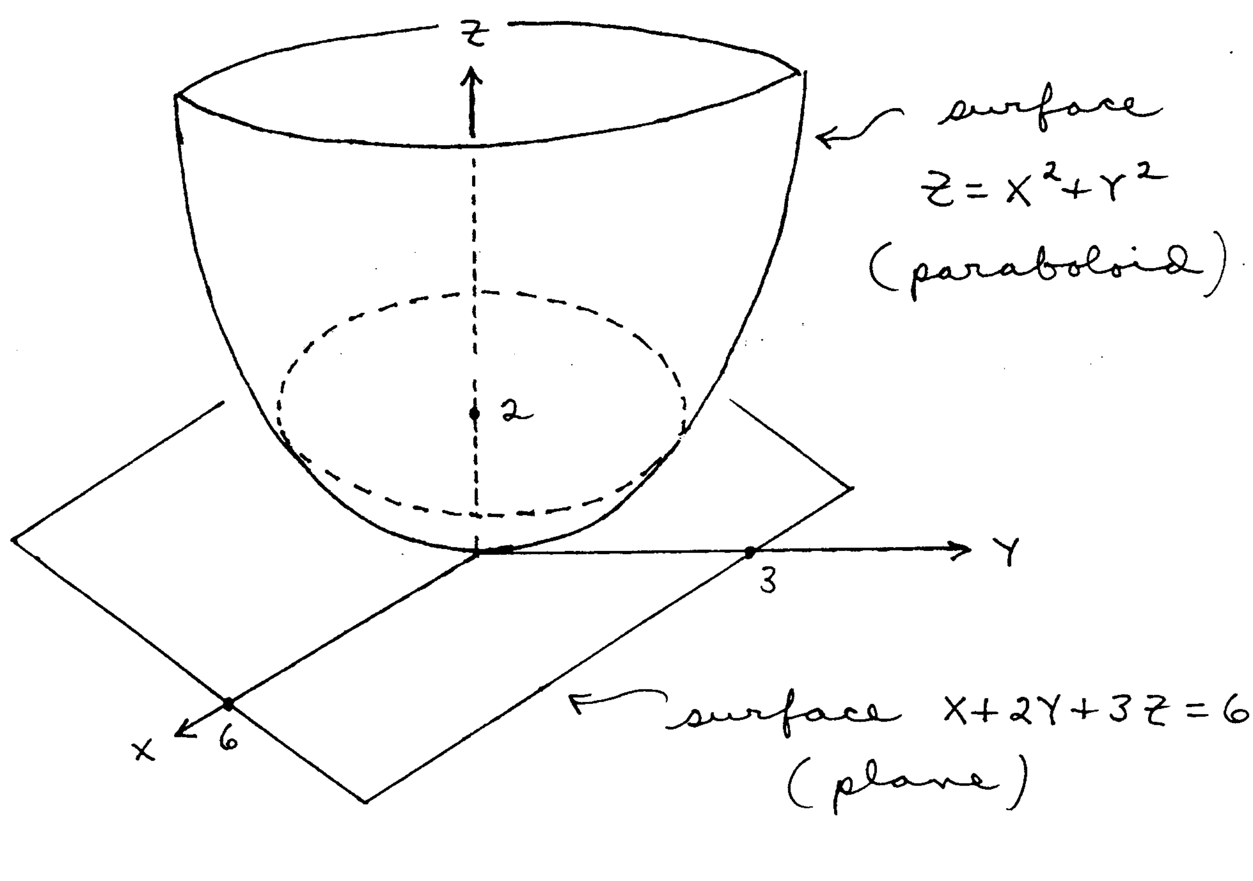
Math 16C
Kouba
__Lagrange Multiplier Problem__

__Ex:__ Minimize and maximize the function $T = 50 + X + 2Y + Z^2$ subject to the two constraints

$$Z = X^2 + Y^2$$

and $$X + 2Y + 3Z = 6 \quad :$$

The two constraints taken together represent the intersection of a plane and a paraboloid. This intersection is represented by the dotted "ring" in the diagram below.



If we assume that the T equation measures the temperature at point (x, y, z), then for this problem we are seeking the points on the "ring" where the temperature is hottest and coldest. The solution using Lagrange multipliers follows :

Let $F = (50 + x + 2y + z^2) - \lambda(z - x^2 - y^2) - \mu(6 - x - 2y - 3z)$

then

$$F_x = 1 + 2\lambda x + \mu = 0$$

$$F_y = 2 + 2\lambda y + 2\mu = 0$$

$$F_z = 2z - \lambda + 3\mu = 0$$

$$F_\lambda = x^2 + y^2 - z = 0$$

$$F_\mu = x + 2y + 3z - 6 = 0 \quad ;$$

solve the first three equations for μ getting

$$\left.\begin{array}{l} \mu = -1 - 2\lambda x \\ \mu = -1 - \lambda y \\ \mu = \frac{-2}{3} z + \frac{1}{3}\lambda \end{array}\right\} \begin{array}{l} 2\lambda x = \lambda y \rightarrow 2\lambda x - \lambda y = 0 \rightarrow \\ \lambda(2x - y) = 0 \rightarrow \boxed{\lambda = 0} \text{ or } \boxed{y = 2x} \end{array};$$

<u>case 1.</u> If $\lambda = 0$ then $\mu = -1$ and $-1 = \frac{-2}{3} z \rightarrow$ $z = \frac{3}{2}$. Substitute into the fourth and fifth equations getting

$$X^2 + Y^2 - \frac{3}{2} = 0$$
$$X + 2Y + \frac{9}{2} - 6 = 0 \rightarrow X = \frac{3}{2} - 2Y$$

$$\left(\frac{3}{2} - 2Y\right)^2 + Y^2 - \frac{3}{2} = 0 \rightarrow$$
$$\frac{9}{4} - 6Y + 4Y^2 + Y^2 - \frac{3}{2} = 0 \rightarrow$$
$$5Y^2 - 6Y + \frac{3}{4} = 0 \rightarrow$$
$$20Y^2 - 24Y + 3 = 0 \rightarrow$$

$$Y = \frac{24 \pm \sqrt{24^2 - 240}}{40} = \frac{24 \pm \sqrt{336}}{40} = 1.06 \text{ or } 0.14$$

so critical points and temperatures are

$X = -0.62$, $Y = 1.06$, $Z = 1.5$ and $T = 53.75°$
$X = 1.22$, $Y = 0.14$, $Z = 1.5$ and $T = 53.75°$.

<u>case 2.</u> If $Y = 2X$ then fourth and fifth equations become

$$\left.\begin{array}{l} X^2 + (2X)^2 - Z = 0 \\ X + 2(2X) + 3Z - 6 = 0 \end{array}\right\} \quad \left.\begin{array}{l} Z = 5X^2 \\ 5X + 3Z = 6 \end{array}\right\} \; 15X^2 + 5X - 6 = 0 \rightarrow$$

$$X = \frac{-5 \pm \sqrt{25 + 360}}{30} = 0.49 \text{ or } -0.82$$

so critical points and temperatures are

$X = -0.82$, $Y = -1.64$, $Z = 3.36$ and $T = 57.19°$
$X = 0.49$, $Y = 0.98$, $Z = 1.20$ and $T = 53.89°$.

The maximum value of T is $57.19°$.
The minimum value of T is $53.75°$.