

Math 21C

Kouba

## The Ratio Test

Ratio Test: Assume  $a_n > 0$  and consider  $\sum_{n=1}^{\infty} a_n$ .

1.) If  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

2.) If  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L > 1$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

3.) If  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ , then the ratio test is inconclusive.

Proof: 1.) Assume  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L < 1$ .  $\frac{L}{\quad} \quad \frac{L+1}{2} \quad 1$

Then there is some integer  $N$  so that

$\frac{a_{n+1}}{a_n} < \frac{L+1}{2} = m$  for all  $n \geq N$ . Thus,

$a_{n+1} < m a_n$  for all  $n \geq N$ , so that

$$a_{N+1} < m a_N,$$

$$a_{N+2} < m a_{N+1} < m^2 a_N,$$

$$a_{N+3} < m a_{N+2} < m^3 a_N,$$

$$a_{N+4} < m a_{N+3} < m^4 a_N, \dots$$

It follows that

$$\sum_{n=N}^{\infty} a_n = a_N + a_{N+1} + a_{N+2} + a_{N+3} + \dots$$

$$\leq a_N + m a_N + m^2 a_N + m^3 a_N + \dots$$

$$= a_N (1 + m + m^2 + m^3 + \dots)$$

$$= a_N \cdot \frac{1}{1-m} \quad (\text{since } -1 < m < 1)$$

$$< \infty.$$

This means that  $\sum_{n=N}^{\infty} a_n$  converges, so that  $\sum_{n=1}^{\infty} a_n$  converges.

2.) Assume  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L > 1$ . Then there

is some integer  $N$  so that  $\frac{a_{n+1}}{a_n} > 1$  for all  $n \geq N$ . Thus,  $a_{n+1} > a_n$  for all  $n \geq N$ , so that

$$0 < a_N < a_{N+1} < a_{N+2} < a_{N+3} \dots$$

This means that  $a_n$  is a positive, increasing sequence for  $n \geq N$ . It follows that  $\lim_{n \rightarrow \infty} a_n \neq 0$  and  $\sum_{n=1}^{\infty} a_n$  diverges by the  $n$ th term test.

QED