Math 16C
Kouba
Taylor Series, Series Summary

I.) Series Tests

A.) (nth term) If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges.

B.) (geometric) $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ for $-1 < r < 1$; otherwise, it diverges.

C.) (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

1.) diverges if $p \le 1$,
2.) converges if $p > 1$, and

$$\frac{1}{(N+1)^p} + \frac{1}{(N+2)^p} + \frac{1}{(N+3)^p} + \cdots < \frac{N^{1-p}}{p-1}.$$

D.) (ratio) $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$

1.) diverges if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$,

2.) converges if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$,

and 3.) is inconclusive if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$.

II.) The Taylor series centered at c for a given function $f(x)$ is

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

where $a_n = \dfrac{f^{(n)}(c)}{n!}$ for $n = 0, 1, 2, 3, \ldots$

Some specific Taylor series centered at $c=0$ are

(1) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$

(2) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$

(3) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

(4) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

III.) The Taylor polynomial of degree n centered at c for a given function $f(x)$ is

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and the absolute error at x is

$$|f(x) - P_n(x)| = |R_n(x)|$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

and z is between x and c.