Math 16C
Kouba
Taylor's Theorem

Given a function $f(x)$ and a constant c, find constants $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ so that

(1) $f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$,

$f(c) = a_0 + a_1(0) + a_2(0) + \cdots = a_0$ so

$a_0 = f(c)$; differentiate (1) then

(2) $f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots + na_n(x-c)^{n-1} + \cdots$,

$f'(c) = a_1 + 2a_2(0) + 3a_3(0) + \cdots = a_1$ so

$a_1 = f'(c)$; differentiate (2) then

(3) $f''(x) = 2a_2 + 3\cdot2a_3(x-c) + 4\cdot3a_4(x-c)^2 + \cdots + n(n-1)a_n(x-c)^{n-2} + \cdots$,

$f''(c) = 2a_2 + 3\cdot2a_3(0) + 4\cdot3a_4(0) + \cdots = 2a_2$ so

$a_2 = \dfrac{f''(c)}{2}$; differentiate (3) then

(4) $f'''(x) = 3\cdot2a_3 + 4\cdot3\cdot2a_4(x-c) + 5\cdot4\cdot3a_5(x-c)^2 + \cdots$,

$f'''(c) = 3\cdot2a_3 + 4\cdot3\cdot2a_4(0) + 5\cdot4\cdot3a_5(0) + \cdots = 3\cdot2a_3$ so

$a_3 = \dfrac{f'''(c)}{3\cdot2}$; differentiate (4) then

(5) $f^{(4)}(x) = 4\cdot3\cdot2a_4 + 5\cdot4\cdot3\cdot2a_5(x-c) + 6\cdot5\cdot4\cdot3a_6(x-c)^2 + \cdots$,

$$f^{(4)}(c) = 4\cdot3\cdot2\, a_4 + 5\cdot4\cdot3\cdot2\, a_5(0) + 6\cdot5\cdot4\cdot3\, a_6(0) + \cdots$$
$$= 4\cdot3\cdot2\, a_4 \qquad \text{so}$$
$$a_4 = \frac{f^{(4)}(c)}{4\cdot3\cdot2} \quad ; \quad \text{continuing this}$$

process, it follows that

$$a_n = \frac{f^{(n)}(c)}{n!} \quad \text{for } n = 1, 2, 3, 4, \ldots \quad .$$

Summarizing, we have :

<u>Taylor's Theorem</u> : If $f(x)$ is a function and c is a constant where

$$(*) \quad f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots ,$$

then

$$\boxed{a_n = \frac{f^{(n)}(c)}{n!}} \quad \text{for } n = 0, 1, 2, 3, 4, \cdots .$$

The series in equation $(*)$ is called the Taylor series centered at c for the function $f(x)$.