Math 17A
Kouba
Differentiating the Natural Logarithm

FACT: $\lim_{z \to 0} (1+z)^{1/z} = e \approx 2.71828$

Let $f(x) = \ln x$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h \to 0} \ln\left(\frac{x}{x} + \frac{h}{x}\right)^{1/h}$$

$$= \lim_{h \to 0} \ln\left(1 + \frac{h}{x}\right)^{1/h}$$

$$= \lim_{h \to 0} \ln\left[\left(1 + \left(\frac{h}{x}\right)\right)^{\frac{1}{\left(\frac{h}{x}\right)}}\right]^{\frac{1}{x}}$$

$$= \ln [e]^{1/x}$$

$$= \frac{1}{x} \cdot \ln e = \frac{1}{x} \cdot (1) = \frac{1}{x}.$$