Math 17A
Kouba
Recursions, Sequences, Fixed Points, and Limits

EXAMPLE : The following recursions and inital values determine a sequence. Find a_n for $n = 1, 2, 3, 4, 5$.

1.) $a_{n+1} = 2a_n + 3, a_0 = -1$

$a_1 = 2a_0 + 3 = 2(-1) + 3 = 1,$
$a_2 = 2a_1 + 3 = 2(1) + 3 = 5,$
$a_3 = 2a_2 + 3 = 2(5) + 3 = 13,$
$a_4 = 2a_3 + 3 = 2(13) + 3 = 29,$
$a_5 = 2a_4 + 3 = 2(29) + 3 = 61$. Hence $\lim_{n\to\infty} a_n = \infty$ (DNE) .

2.) $a_{n+1} = 2a_n + 3, a_0 = -3$

$a_1 = 2a_0 + 3 = 2(-3) + 3 = -3,$
$a_2 = 2a_1 + 3 = 2(-3) + 3 = -3,$
$a_3 = 2a_2 + 3 = 2(-3) + 3 = -3,$
$a_4 = 2a_3 + 3 = 2(-3) + 3 = -3,$
$a_5 = 2a_4 + 3 = 2(-3) + 3 = -3$. Hence $\lim_{n\to\infty} a_n = -3$.

DEFINITION : Let $a_{n+1} = f(a_n)$, $a_0 = L$, for $n = 1, 2, 3, 4, \cdots$ be a recursion and initial value which determines a sequence. The initial value L is called a fixed point for the recursion if all successive values of a_n are equal to L, i.e., if $L = f(L)$.

NOTE :

I.) The number -3 is a fixed point for the previous example.

II.) The initial value is sometimes critical in determining if the sequence converges or diverges.

III.) A fixed point represents a potential limit for the sequence generated by the recursion and its initial value.

IV.) Every limit of an associated sequence is a fixed point for the recursion.

EXAMPLE : Find all fixed points for each recursion.

1.) $a_{n+1} = (1/2)a_n - (3/4)$

2.) $a_{n+1} = \dfrac{2}{a_n - 1}$

3.) $a_{n+1} = \dfrac{a_n^2}{a_n^2 - 12}$