Section 5.3

1.) $y = (2-x)^2$ for $-2 \le x \le 3$

$\xrightarrow{D} y' = 2(2-x)(-1) = 2(x-2) = 0$

$\xrightarrow{D} y'' = 2 > 0$

y'': $+ \quad + \quad +$ on $x=-2$ to $x=3$

f': $- \quad 0 \quad +$ on $x=-2$, $x=2$, $x=3$

$x=-2$, $y=16$: abs. max.

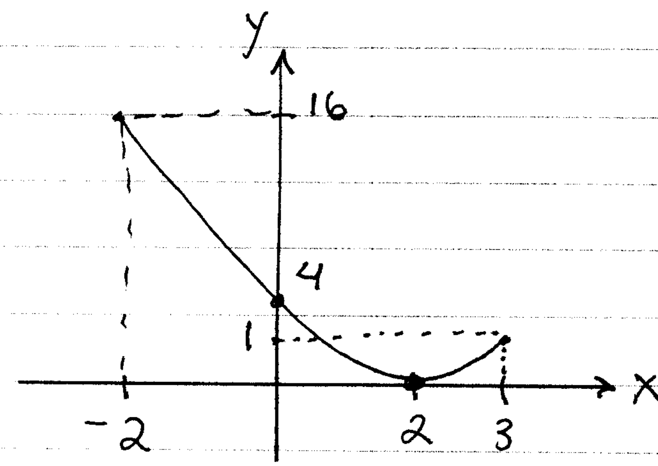
$x=2$, $y=0$: abs. min.

$x=3$, $y=1$: rel. max.

$x=0: y=4$

$y=0: (2-x)^2 = 0$

$\rightarrow x = 2$



2.) $y = \sqrt{x-1}$ for $1 \le x \le 2$

$\xrightarrow{D} y' = \frac{1}{2}(x-1)^{-1/2}(1) = \frac{1}{2\sqrt{x-1}} = 0$ (impossible)

NO

$\xrightarrow{D} y'' = \frac{-1}{4}(x-1)^{-3/2}$

$= \dfrac{-1}{4(x-1)^{3/2}}$

$= 0$ (impossible)

f': NO $\quad + \quad + \quad +$

abs. min. $\begin{cases} x=1 \\ y=0 \end{cases}$ abs. max. $\begin{cases} x=2 \\ y=1 \end{cases}$

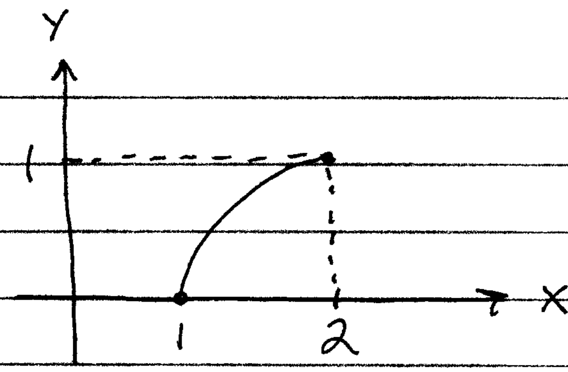
f'': NO $\quad - \quad - \quad -$ on $x=1$ to $x=2$

$x=0$: NO

$y=0 : x=1$

y is $\uparrow$ for $1<x<2$,

y is $\cap$ for $1<x<2$;



7.) $y=(x-1)^3+1$ Domain: all x-values

$\xrightarrow{D} y'=3(x-1)^2=0$

$+ \quad 0 \quad +$ y'

$x=1$

$\xrightarrow{D} y''=6(x-1)=0$

$- \quad 0 \quad +$ y''

$x=0: y=0$

$y=0: (x-1)^3+1=0$

infl. pt. $\begin{cases} x=1 \\ y=1 \end{cases}$

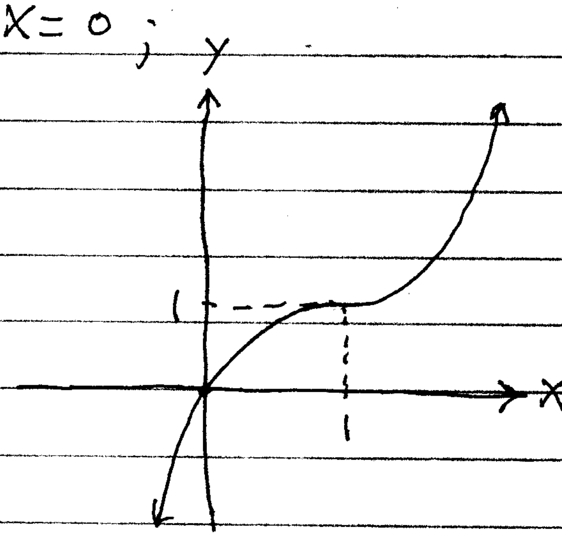
$\rightarrow (x-1)^3=-1$

$\rightarrow x-1=(-1)^{1/3}=-1 \rightarrow x=0;$

y is $\uparrow$ for $x<1$, $x>1$,

y is $\cup$ for $x>1$,

y is $\cap$ for $x<1$,



12.) $y = e^{-\frac{1}{4}x^2}$ Domain: all x-values

$\xrightarrow{D} y' = e^{-\frac{1}{4}x^2}\cdot\frac{-1}{2}x = \frac{-x}{2e^{\frac{1}{4}x^2}} = 0$

$\begin{array}{ccccc} + & & 0 & & - \end{array}$ y'

abs. max. $\begin{cases} x=0 \\ Y=1 \end{cases}$

$\xrightarrow{D} y'' = \frac{e^{\frac{1}{4}x^2}(-1)-(-x)\cdot e^{\frac{1}{4}x^2}\cdot\frac{1}{2}x}{\left(e^{\frac{1}{4}x^2}\right)^2}\cdot\frac{1}{2}$

$= \frac{\cancel{e^{\frac{1}{4}x^2}}\cdot\left[\frac{1}{2}x^2-1\right]}{\left(e^{\frac{1}{4}x^2}\right)^2}\cdot\frac{1}{2} = \frac{\frac{1}{2}x^2-1}{2e^{\frac{1}{4}x^2}} = 0 \longrightarrow$

$\frac{1}{2}x^2-1=0 \longrightarrow x^2=2 \longrightarrow x=\pm\sqrt{2}$

$\longrightarrow$

$\begin{array}{ccccc} + & 0 & - & 0 & + \\ & x=-\sqrt{2} & & x=\sqrt{2} & \\ & Y=e^{-1/2} & & Y=e^{-1/2} & \end{array}$ y''

← infl pt. ↗

$x=0$: $Y=0$,

$Y=0$: impossible;

y is ↑ for $x<0$,

y is ↓ for $x>0$,

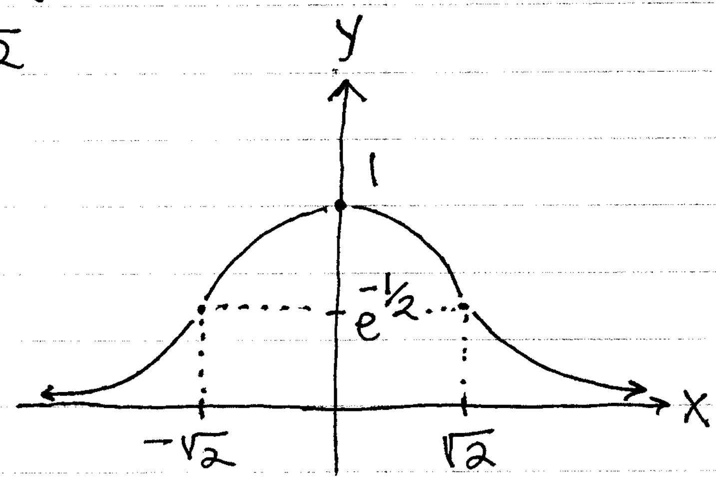
y is $\cup$ for $x<-\sqrt{2}$, $x>\sqrt{2}$

y is $\cap$ for $-\sqrt{2}<x<\sqrt{2}$

$\lim\limits_{x\to\pm\infty} e^{-\frac{1}{4}x^2} = e^{-\infty}$

$= \frac{1}{e^{\infty}} = 0$ so

H.A. is $y=0$



14.) $Y = X^2 - X^3$, Domain : all x-values

$\xrightarrow{D}$ $Y' = 2X - 3X^2 = X(2-3X) = 0 \rightarrow X=0,\ X=\frac{2}{3}$;

$\xrightarrow{D}$ $Y'' = 2 - 6X$

$= 2(1-3X) = 0$

$\rightarrow X = 1/3$;

$-\quad 0 \quad +\quad 0 \quad -$ $\quad Y'$

rel. min. $\begin{cases} X=0 \\ Y=0 \end{cases}$ $\quad$ $\left.\begin{matrix} X=\frac{2}{3} \\ Y=4/27 \end{matrix}\right\}$ rel. max.

Y is $\uparrow$ for $0 < X < 2/3$,

Y is $\downarrow$ for $X<0$, $X > 2/3$,

Y is $\cup$ for $X < 1/3$,

Y is $\cap$ for $X > 1/3$;

$+\quad 0 \quad -$ $\quad Y''$

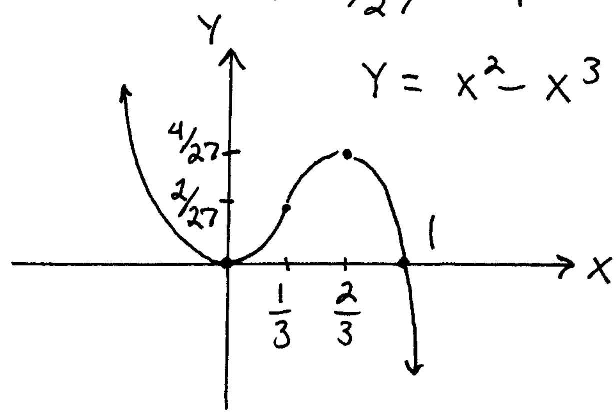
$\left.\begin{matrix} X=1/3 \\ Y=2/27 \end{matrix}\right\}$ infl. pt.

$X=0$: $Y=0$

$Y=0$: $X^2 - X^3 = 0$

$\rightarrow X^2(1-X) = 0$

$\rightarrow X=0,\ X=1$



16.) $Y = \sqrt{1+X^2}$ Domain : all x-values

$\xrightarrow{D}$ $Y' = \frac{1}{2}(1+X^2)^{-1/2} \cdot 2X = \frac{X}{\sqrt{1+X^2}} = 0 \rightarrow X=0$;

$\xrightarrow{D}$ $Y'' = \dfrac{\sqrt{1+X^2}\cdot(1) - X\cdot\frac{1}{\cancel{2}}(1+X^2)^{-1/2}\cdot\cancel{2}X}{1+X^2}$

$-\quad 0 \quad +$ $\quad Y'$

$\left.\begin{matrix} X=0 \\ Y=1 \end{matrix}\right\}$ abs. min.

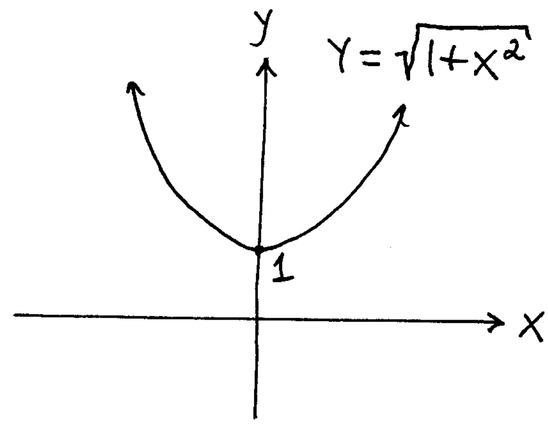
$= \left(\dfrac{\sqrt{1+X^2}}{1} - \dfrac{X^2}{\sqrt{1+X^2}}\right)\cdot\dfrac{1}{1+X^2}$

$= \dfrac{(1+X^2) - X^2}{\sqrt{1+X^2}}\cdot\dfrac{1}{1+X^2}$

$+\quad +\quad +$ $\quad Y''$

$= \dfrac{1}{(1+X^2)^{3/2}}$;

Y is $\uparrow$ for $x > 0$,
Y is $\downarrow$ for $x < 0$,
Y is $\cup$ for all x-values;
$x = 0: \; Y = 0$
$Y = 0: \; \sqrt{1+x^2} = 0$ (No)



22.) $f(x) = xe^{-x}$; Domain: all x-values

$\xrightarrow{D} f'(x) = x \cdot -e^{-x} + (1)e^{-x} = e^{-x}(1-x) = 0 \longrightarrow$

$x = 1$;

$+ \quad 0 \quad -$ $\quad Y'$

abs. max. $\begin{cases} x = 1 \\ Y = 1/e \end{cases}$

$\xrightarrow{D} Y'' = e^{-x}(-1) + -e^{-x}(1-x)$
$= e^{-x}(-1-1+x)$
$= e^{-x}(x-2) = 0 \longrightarrow$

$x = 2$;

$- \quad 0 \quad +$ $\quad Y''$

infl. pt. $\begin{cases} x = 2 \\ Y = 2/e^2 \end{cases}$

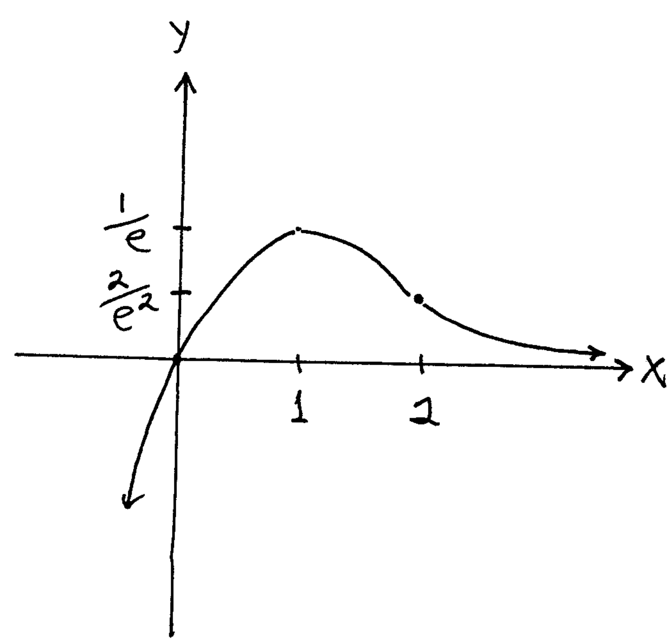
Y is $\uparrow$ for $x < 1$,
Y is $\downarrow$ for $x > 1$,
Y is $\cup$ for $x > 2$,
Y is $\cap$ for $x < 2$;

$x = 0: \; Y = 0$
$Y = 0: \; \dfrac{x}{e^x} = 0 \longrightarrow x = 0$;

$$\lim_{x \to \infty} xe^{-x} = \lim_{x \to \infty} \frac{x}{e^x} = 0$$

so H.A. is $\boxed{Y = 0}$;

$$\lim_{x \to -\infty} xe^{-x} = -\infty \cdot \infty = -\infty.$$



28.) $Y = X^4 - 2X^2$, Domain: all X-values

$\xrightarrow{D} Y' = 4X^3 - 4X = 4X(X^2-1) = 4X(X-1)(X+1) = 0$

$\rightarrow X = 0,\ X = 1,\ X = -1$;

	$X=-1$		$X=0$		$X=1$		
$-$	0	$+$	0	$-$	0	$+$	Y'
	$Y=-1$		$Y=0$		$Y=-1$		
	abs. min.		rel. max.		abs. min.		

$\xrightarrow{D} Y'' = 12X^2 - 4$

$= 4(3X^2 - 1)$

$= 4(\sqrt{3}\cdot X - 1)(\sqrt{3}\cdot X + 1) = 0$

$\rightarrow X = \frac{1}{\sqrt{3}},\ X = \frac{-1}{\sqrt{3}}$;

$+$	0	$-$	0	$+$	Y''
	$X = -1/\sqrt{3}$, $Y = -5/9$ (infl. pt.)		$X = 1/\sqrt{3}$, $Y = -5/9$ (infl. pt.)		

Y is $\uparrow$ for $-1 < X < 0$, $X > 1$,

Y is $\downarrow$ for $X < -1$, $0 < X < 1$,

Y is $\cup$ for $X < -1/\sqrt{3}$, $X > 1/\sqrt{3}$,

Y is $\cap$ for $\frac{-1}{\sqrt{3}} < X < \frac{1}{\sqrt{3}}$;

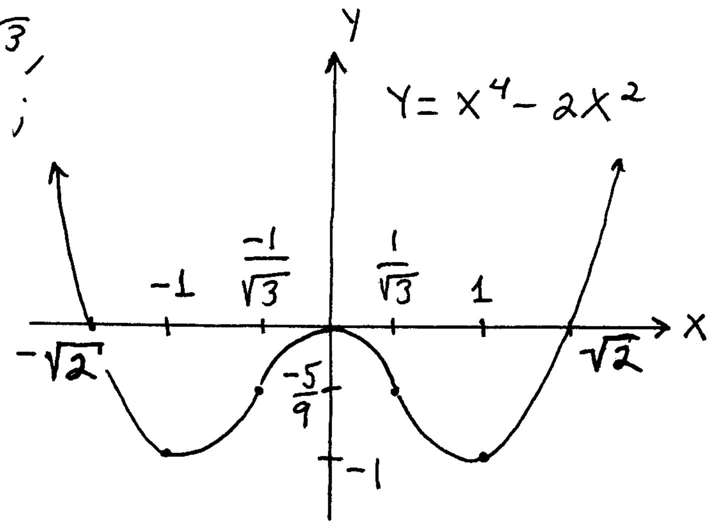
$X = 0:\ Y = 0$

$Y = 0:\ X^4 - 2X^2 = 0$

$\rightarrow X^2(X^2 - 2) = 0$

$\rightarrow X^2(X - \sqrt{2})(X + \sqrt{2}) = 0$

$\rightarrow X = \sqrt{2},\ X = -\sqrt{2}$



33.) $Y = \dfrac{X^2 - 1}{X^2 + 1}$, Domain: all X-values

$\xrightarrow{D} Y' = \dfrac{(X^2+1)(2X) - (X^2-1)(2X)}{(X^2+1)^2} = \dfrac{2X((X^2+1) - (X^2-1))}{(X^2+1)^2}$

$= \dfrac{4X}{(X^2+1)^2} = 0 \longrightarrow X = 0$;

$-$	0	$+$	Y'
	$X=0$, $Y=-1$ abs. min.		

$\xrightarrow{D} Y'' = \dfrac{(X^2+1)^2\cdot(4) - 4X\cdot 2(X^2+1)(2X)}{(X^2+1)^4}$

$$= \frac{4\cancel{(x^2+1)}\cdot[(x^2+1)-4x^2]}{(x^2+1)^{\cancel{4}\,3}} = \frac{4[1-3x^2]}{(x^2+1)^3} = 0$$

$\longrightarrow x = \frac{1}{\sqrt{3}},\ x = \frac{-1}{\sqrt{3}}$;

$$\begin{array}{ccccccc} - & & 0 & + & 0 & & - \end{array} \quad y''$$

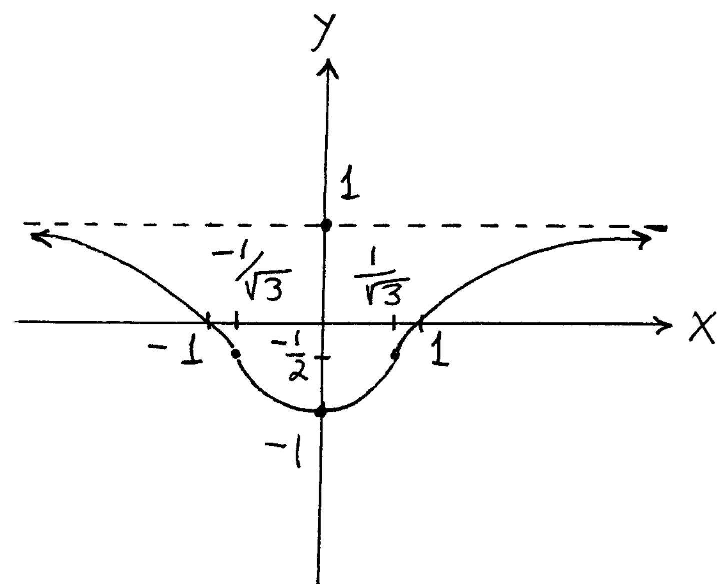
infl. pt. $\left\{\begin{array}{l} x = -1/\sqrt{3} \\ y = -1/2 \end{array}\right.$ $\quad$ $\left.\begin{array}{l} x = 1/\sqrt{3} \\ y = -1/2 \end{array}\right\}$ infl. pt.

y is $\uparrow$ for $x > 0$,

y is $\downarrow$ for $x < 0$,

y is $\cup$ for $\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$,

y is $\cap$ for $x < \frac{-1}{\sqrt{3}}$, $x > \frac{1}{\sqrt{3}}$;



$x = 0$: $y = -1$

$y = 0$: $\frac{x^2-1}{x^2+1} = 0$

$\rightarrow x^2-1 = (x-1)(x+1) = 0$

$\rightarrow x = 1,\ x = -1$;

$$\lim_{x\to\pm\infty} \frac{x^2-1}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\pm\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1-0}{1+0} = 1 \text{ so}$$

H.A. is $\boxed{y = 1}$.

36.) $f(x) = \frac{-2}{x^2-1} = \frac{-2}{(x-1)(x+1)}$, Domain: all $x \neq \pm 1$

$\xrightarrow{D} f'(x) = \frac{(x^2-1)(0)-(-2)(2x)}{(x^2-1)^2} = \frac{4x}{(x^2-1)^2} = 0 \rightarrow$

$x = 0$;

$$\begin{array}{ccccccccc} - & \overset{NO}{\vdots} & - & 0 & + & \overset{NO}{\vdots} & + \\ & x=-1 & & x=0 & & x=1 & \end{array} \quad y'$$

$y = 2$

rel. min.

$$\xrightarrow{D} f''(x) = \frac{(x^2-1)^2(4) - 4x\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}$$

$$= \frac{4\cancel{(x^2-1)}\cdot\left[(x^2-1)-4x^2\right]}{(x^2-1)^{\cancel{4}\,3}} = \frac{4\left[-1-3x^2\right]}{(x^2-1)^3}$$

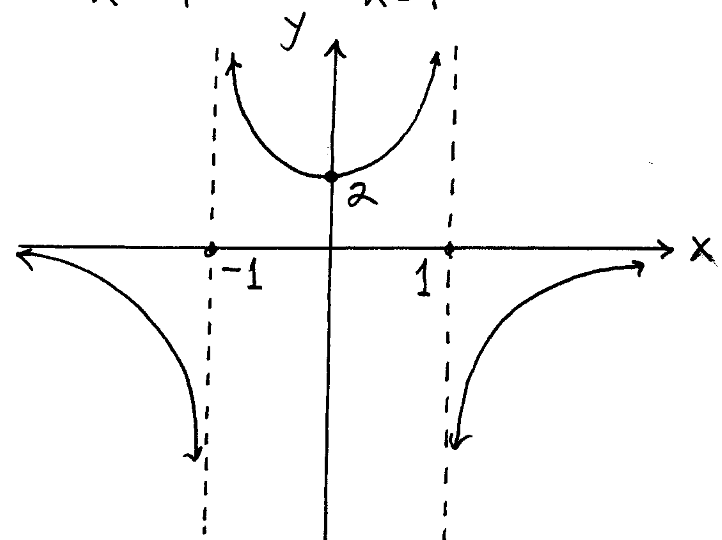
$$= \frac{-4\left[1+3x^2\right]}{(x^2-1)^3} ;$$

−	NO	+	NO	−	Y''
	$x=-1$		$x=1$		

Y is ↑ for $0<x<1$, $x>1$,
Y is ↓ for $x<-1$, $-1<x<0$,
Y is ∪ for $-1<x<1$,
Y is ∩ for $x<-1$, $x>1$;

$$\lim_{x\to\pm\infty} \frac{-2}{x^2-1} = \text{"}\frac{-2}{\infty}\text{"} = 0$$

so H.A. is $\boxed{Y=0}$;



$$\lim_{x\to 1^+} \frac{-2}{(x-1)(x+1)} = \frac{-2}{(0^+)(2)} = \text{"}\frac{-2}{0^+}\text{"} = -\infty, \text{ so V.A. is } \boxed{x=1};$$

$$\lim_{x\to 1^-} \frac{-2}{(x-1)(x+1)} = \frac{-2}{(0^-)(2)} = \text{"}\frac{-2}{0^-}\text{"} = +\infty, \text{ so V.A. is } \boxed{x=1};$$

$$\lim_{x\to -1^+} \frac{-2}{(x-1)(x+1)} = \frac{-2}{(-2)(0^+)} = \text{"}\frac{1}{0^+}\text{"} = +\infty \text{ so V.A. is } \boxed{x=-1};$$

$$\lim_{x\to -1^-} \frac{-2}{(x-1)(x+1)} = \frac{-2}{(-2)(0^-)} = \text{"}\frac{1}{0^-}\text{"} = -\infty \text{ so V.A. is } \boxed{x=-1}.$$

$x=0$: $Y=2$
$Y=0$: $\frac{-2}{x^2-1}=0 \longrightarrow$ (NO)

37.) $f(x) = \dfrac{2x^2-6}{x+2}$; Domain: all $x \neq -2$

$\xrightarrow{D} f'(x) = \dfrac{(x+2)(4x) - (2x^2-6)(1)}{(x+2)^2}$

$= \dfrac{4x^2+8x-2x^2+6}{(x+2)^2} = \dfrac{2x^2+8x+6}{(x+2)^2}$

$= \dfrac{2(x^2+4x+3)}{(x+2)^2} = \dfrac{2(x+1)(x+3)}{(x+2)^2} = 0 \longrightarrow$

$x=-1,\ x=-3$;

	NO		
+ 0 −	⋮	− 0 +	f'
$x=-3$	$x=-2$	$x=-1$	
$\underbrace{y=-12}$		$\underbrace{y=-4}$	
rel. max.		rel. min.	

$\xrightarrow{D} f''(x) = \dfrac{(x+2)^2(4x+8) - (2x^2+8x+6)\cdot 2(x+2)}{(x+2)^4}$

$= \dfrac{2\cancel{(x+2)}\cdot[(x+2)(2x+4) - (2x^2+8x+6)]}{(x+2)^{\cancel{4}\,3}}$

$= \dfrac{2\cdot[\cancel{2x^2}+\cancel{8x}+8-\cancel{2x^2}-\cancel{8x}-6]}{(x+2)^3}$

$= \dfrac{4}{(x+2)^3}$;

	NO		
−	⋮	+	f''
	$x=-2$		

$x=0:\ y=-3$

$y=0:\ \dfrac{2x^2-6}{x+2} = 0 \longrightarrow 2x^2-6=0 \longrightarrow$

$2(x-\sqrt{3})(x+\sqrt{3}) = 0 \longrightarrow x=\sqrt{3},\ x=-\sqrt{3}$;

$\lim\limits_{x\to\infty} \dfrac{2x^2-6}{x+2}\cdot\dfrac{1/x}{1/x} = \lim\limits_{x\to\infty} \dfrac{2x-\frac{6}{x}}{1+\frac{2}{x}} = \dfrac{\infty-0}{1+0} = \infty$;

$$\lim_{x\to-\infty} \frac{2x^2-6}{x+2}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty}\frac{2x-\frac{6}{x}}{1+\frac{2}{x}}$$

$$= \frac{-\infty-0}{1+0} = -\infty \, ;$$

$$\lim_{x\to-2^+}\frac{2x^2-6}{x+2} = \frac{"2"}{0^+} = \infty \text{ so V.A. is } \boxed{x=-2};$$

$$\lim_{x\to-2^-}\frac{2x^2-6}{x+2} = \frac{"2"}{0^-} = -\infty \text{ so V.A. is } \boxed{x=-2};$$

$$\begin{array}{r l}
 & 2x-4 \\
x+2 \,\big)\!\! & \overline{2x^2-6} \\
 & \underline{2x^2+4x} \\
 & \quad -4x-6 \\
 & -\underline{(-4x-8)} \\
 & \qquad\quad 2
\end{array}$$

so $f(x) = \frac{2x^2-6}{x+2}$

$= 2x-4+\frac{2}{x+2}$

so $\boxed{y=2x-4}$ is tilted asymptote ;