Section 2.2

2.) $a_n = 3n^2$: 0, 3, 12, 27, 48, 75
n : 0 1 2 3 4 5

9.) $a_n = (-1)^n n$: 0, −1, 2, −3, 4, −5
n : 0 1 2 3 4 5

10.) $a_n = \frac{(-1)^n}{(n+1)^2}$: 1, $\frac{-1}{4}$, $\frac{1}{9}$, $\frac{-1}{16}$, $\frac{1}{25}$, $\frac{-1}{36}$
n : 0 1 2 3 4 5

19.) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \frac{1}{64}, \frac{1}{81}, \ldots$

20.) $-1, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16}, \frac{-1}{25}, \frac{1}{36}, \frac{-1}{49}, \frac{1}{64}, \frac{-1}{81}, \ldots$

21.) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \ldots$

22.) $\frac{1}{5}, \frac{4}{10}, \frac{9}{17}, \frac{16}{26}, \frac{25}{37}, \frac{36}{50}, \frac{49}{65}, \frac{64}{82}, \frac{81}{101}$

26.) a_n : 0, 2, 4, 6, 8, ...
a_n : 0, 2·1, 2·2, 2·3, 2·4, ..., $\boxed{2n}$
n : 0 1 2 3 4 ... n

27.) a_n : 1, 2, 4, 8, 16, ...
a_n : $2^0, 2^1, 2^2, 2^3, 2^4, \ldots,$ $\boxed{2^n}$
n : 0 1 2 3 4 ... n

28.) $a_n: 1, 3, 5, 7, 9, \ldots$

$a_n: 0+1, 2+1, 2\cdot2+1, 2\cdot3+1, 2\cdot4+1, \ldots$ $\boxed{2n+1}$

$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n$

29.) $a_n: 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

$a_n: \frac{1}{3^0}, \frac{1}{3^1}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \ldots$ $\boxed{\frac{1}{3^n}}$

$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n$

30.) $a_n: \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \ldots$

$a_n: \frac{1}{2+1}, \frac{2}{2\cdot2+1}, \frac{3}{2\cdot3+1}, \frac{4}{2\cdot4+1}, \frac{5}{2\cdot5+1}, \ldots$ $\boxed{\frac{n+1}{2(n+1)+1}}$

$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n$

31.) $a_n: -1, 2, -3, 4, -5, \ldots$

$a_n: (-1)^1\cdot1, (-1)^2\cdot2, (-1)^3\cdot3, (-1)^4\cdot4, (-1)^5\cdot5, \ldots,$ $\boxed{(-1)^{n+1}(n+1)}$

$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n$

34.) $a_n: \frac{1}{2}, \frac{-1}{8}, \frac{1}{18}, \frac{-1}{32}, \frac{1}{50}, \ldots$

$a_n: \frac{(-1)^0}{2\cdot1^2}, \frac{(-1)^1}{2\cdot2^2}, \frac{(-1)^2}{2\cdot3^2}, \frac{(-1)^3}{2\cdot4^2}, \frac{(-1)^4}{2\cdot5^2}, \ldots$ $\boxed{\frac{(-1)^n}{2\cdot(n+1)^2}}$

$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n$

38.) $a_n = \frac{2}{n+1}$: $2, 1, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \ldots$

$$\lim_{n\to\infty} \frac{2}{n+1} = \text{"}\frac{2}{\infty}\text{"} = 0$$

39.) $a_n = \frac{n}{n+1}$: $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

$$\lim_{n\to\infty} \frac{n}{n+1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{1+0} = 1$$

43.) $a_n = \frac{(-1)^n}{n+1}$: $1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \ldots$

$\frac{-1}{n+1} \leq \frac{(-1)^n}{n+1} \leq \frac{+1}{n+1}$ and

$\lim_{n\to\infty} \frac{-1}{n+1} = 0 = \lim_{n\to\infty} \frac{1}{n+1}$, so

$\lim_{n\to\infty} \frac{(-1)^n}{n+1} = 0$ by Squeeze Principle

45.) $a_n = \frac{n^2}{n+1}$: $0, 1, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$

$$\lim_{n\to\infty} \frac{n^2}{n+1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{n^2}{n+1} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{n}{1+1/n} = \frac{\infty}{1+0} = \infty \quad \text{(DNE)}$$

49.) $a_n = 2^n$: $1, 2, 4, 8, 16, \ldots$

$\lim_{n\to\infty} 2^n = \infty$ (DNE)

50.) $a_n = \left(\frac{1}{2}\right)^n$: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

54.) $\lim_{n\to\infty} \frac{1}{n} = 0$ so

$|a_n - a| < \varepsilon$ iff $\left|\frac{1}{n} - 0\right| < 0.02$

iff $\frac{1}{n} < \frac{2}{100} = \frac{1}{50}$

iff $n > 50$; let $N = 50$.

61.) $\lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$ so

$|a_n - a| < \varepsilon$ iff $\left|\frac{n}{n+1} - 1\right| < 0.01$

iff $\left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| < \frac{1}{100}$

iff $\left|\frac{-1}{n+1}\right| < \frac{1}{100}$

iff $\frac{1}{n+1} < \frac{1}{100}$

iff $n+1 > 100$

iff $n > 99$; let $N = 99$.

66.) Prove: $\lim_{n\to\infty} \frac{1}{n+1} = 0$. Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $\left|\frac{1}{n+1} - 0\right| < \varepsilon$. Then

$\left|\frac{1}{n+1} - 0\right| < \varepsilon$ iff $\frac{1}{n+1} < \varepsilon$

iff $n+1 > \frac{1}{\varepsilon}$

iff $n > \frac{1}{\varepsilon} - 1$; let N be any integer $\geq \frac{1}{\varepsilon} - 1$.

68.) Prove: $\lim_{n \to \infty} \frac{1}{n^2+1} = 0$. Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $\left| \frac{1}{n^2+1} - 0 \right| < \varepsilon$. Then

$\left| \frac{1}{n^2+1} - 0 \right| < \varepsilon$ iff $\frac{1}{n^2+1} < \varepsilon$

iff $n^2 + 1 > \frac{1}{\varepsilon}$

iff $n^2 > \frac{1}{\varepsilon} - 1$

(Note: If $\varepsilon > 1$, then $\frac{1}{\varepsilon} - 1 < 0$, so ANY $N \geq 0$ works.)

iff $n > \sqrt{\frac{1}{\varepsilon} - 1}$ (assume $\varepsilon \leq 1$.);

let N be any integer $\geq \sqrt{\frac{1}{\varepsilon} - 1}$.

70.) Prove: $\lim_{n \to \infty} \frac{n}{n+1} = 1$. Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $\left| \frac{n}{n+1} - 1 \right| < \varepsilon$. Then

$\left| \frac{n}{n+1} - 1 \right| < \varepsilon$ iff $\left| \frac{n}{n+1} - \frac{n+1}{n+1} \right| < \varepsilon$

iff $\left| \frac{-1}{n+1} \right| < \varepsilon$

iff $\frac{1}{n+1} < \varepsilon$

iff $n+1 > \frac{1}{\varepsilon}$

iff $n > \frac{1}{\varepsilon} - 1$

(note: If $\varepsilon > 1$, then $\frac{1}{\varepsilon} - 1 < 0$, so ANY $N \geq 0$ works.);

Assume $\varepsilon \leq 1$ and let N be any integer $\geq \frac{1}{\varepsilon} - 1$.

72.) $\lim_{n\to\infty} \left(\frac{2}{n} - \frac{1}{n^2+1}\right) = \lim_{n\to\infty} \frac{2}{n} - \lim_{n\to\infty} \frac{1}{n^2+1}$

$= \frac{2}{\infty} - \frac{1}{\infty} = 0 - 0 = 0$

75.) $\lim_{n\to\infty} \left(\frac{n^2+1}{n^2}\right) = \lim_{n\to\infty} \left(\frac{n^2}{n^2} + \frac{1}{n^2}\right)$

$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n^2} = 1 + \frac{1}{\infty} = 1 + 0 = 1$

78.) $\lim_{n\to\infty} \left(\frac{n+2}{n^2-4}\right) = \lim_{n\to\infty} \frac{\cancel{n+2}}{(\cancel{n+2})(n-2)}$

$= \lim_{n\to\infty} \frac{1}{n-2} = \frac{1}{\infty} = 0$

81.) $\lim_{n\to\infty} \frac{n+2^{-n}}{n} = \lim_{n\to\infty} \left(\frac{n}{n} + \frac{2^{-n}}{n}\right)$

$= \lim_{n\to\infty} \left(1 + \frac{1}{n \cdot 2^n}\right) = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n \cdot 2^n}$

$$= 1 + \frac{1}{\infty} = 1 + 0 = 1$$

83.) $a_{n+1} = 2a_n$, $a_0 = 1$:

$a_1 = 2a_0 = 2(1) = 2$,
$a_2 = 2a_1 = 2(2) = 4$,
$a_3 = 2a_2 = 2(4) = 8$,
$a_4 = 2a_3 = 2(8) = 16$,
$a_5 = 2a_4 = 2(16) = 32$

88.) $a_{n+1} = 4 - 2a_n$, $a_0 = 4/3$:

$a_1 = 4 - 2a_0 = 4 - 2(4/3) = 4 - \frac{8}{3} = 4/3$,
$a_2 = 4 - 2a_1 = 4 - 2\left(\frac{4}{3}\right) = \frac{4}{3}$,
$a_3 = 4/3$, $a_4 = 4/3$, $a_5 = 4/3$

91.) $a_{n+1} = a_n + \frac{1}{a_n}$, $a_0 = 1$:

$a_1 = a_0 + \frac{1}{a_0} = 1 + \frac{1}{1} = 2$,

$a_2 = a_1 + \frac{1}{a_1} = 2 + \frac{1}{2} = \frac{5}{2}$,

$a_3 = a_2 + \frac{1}{a_2} = \frac{5}{2} + 1/5/2 = \frac{5}{2} + \frac{2}{5} = \frac{29}{10}$,

$a_4 = a_3 + \frac{1}{a_3} = \frac{29}{10} + \frac{1}{\frac{29}{10}} = \frac{29}{10} + \frac{10}{29} = \frac{941}{290}$,

$a_5 = a_4 + \frac{1}{a_4} = \frac{941}{290} + \frac{1}{\frac{941}{290}} = \frac{941}{290} + \frac{290}{941} = \frac{969,581}{272,890}$

93.) assume $\lim_{n\to\infty} a_n = L$; then

$$a_{n+1} = \frac{1}{2}a_n + 2 \longrightarrow (n\to\infty) \quad L = \frac{1}{2}L + 2 \longrightarrow$$
$$\frac{1}{2}L = 2 \longrightarrow \quad L = 4$$

96.) assume $\lim_{n\to\infty} a_n = L$; then

$$a_{n+1} = \frac{-1}{3}a_n + \frac{1}{4} \longrightarrow (n\to\infty) \quad L = \frac{-1}{3}L + \frac{1}{4} \longrightarrow$$
$$\frac{4}{3}L = \frac{1}{4} \longrightarrow \quad L = \frac{1}{4} \div \frac{4}{3} = \frac{1}{4}\cdot\frac{3}{4} = \frac{3}{16}$$

100.) assume $\lim_{n\to\infty} a_n = L$; then

$$a_{n+1} = \frac{3}{a_n - 2} \longrightarrow (n\to\infty) \quad L = \frac{3}{L-2} \longrightarrow$$

$$L^2 - 2L = 3 \longrightarrow L^2 - 2L - 3 = 0 \longrightarrow$$
$$(L-3)(L+1) = 0 \longrightarrow \quad L = 3 \text{ or } L = -1$$

105.) assume $\lim_{n\to\infty} a_n = L$; then

$$a_{n+1} = \sqrt{2a_n} \longrightarrow (n\to\infty) \quad L = \sqrt{2L} \longrightarrow$$
$$L^2 = 2L \longrightarrow L^2 - 2L = 0 \longrightarrow$$
$$L(L-2) = 0 \longrightarrow$$

$L = 0$ or $L = 2$;

the table suggests the limit is 2 for $a_0 = 1$.

n	a{n}
0	1.00
1	1.41
2	1.68
3	1.83
4	1.92
5	1.96
6	1.98
7	1.99
8	1.99
9	2.00
10	2.00

106.) If $a_{n+1} = \sqrt{2a_n}$ and $a_0 = 0$, then

$a_1 = \sqrt{2a_0} = \sqrt{2(0)} = \sqrt{0} = 0$,

$a_2 = \sqrt{2a_1} = \sqrt{2(0)} = \sqrt{0} = 0$,

$a_3 = 0$, $a_4 = 0$, $a_5 = 0$, ... so

$\lim_{n\to\infty} a_n = 0$ if $a_0 = 0$

109.) assume $\lim_{n\to\infty} a_n = L$; then

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{4}{a_n}\right) \to (n\to\infty)\ L = \frac{1}{2}\left(L + \frac{4}{L}\right)$

$\to 2L = L + \frac{4}{L} \to 2L^2 = L^2 + 4 \to L^2 = 4$

$\to L = 2$ or $L = -2$;

the table suggests the limit is 2 if $a_0 = 1$.

n	a{n}
0	1.00000
1	2.50000
2	2.05000
3	2.00061
4	2.00000
5	2.00000

ε, N Proof Worksheet

Reminder: 1.) Start the proof with 3 sentences.
2.) The middle of the proof consists of your "iff" statements.
3.) Finish the proof with 2 sentences.

1.) $\lim_{n\to\infty} \frac{500}{n^3+2} = \frac{500}{\infty} = 0$.

Proof: Let $\varepsilon > 0$ be given. Find an integer N so that if $n > N$, then $|a_n - L| < \varepsilon$. Begin with $|a_n - L| < \varepsilon$ and solve for n.

Then $|a_n - L| < \varepsilon$

iff $\left| \frac{500}{n^3+2} - 0 \right| < \varepsilon$

iff $\frac{500}{n^3+2} < \varepsilon$ $(n^3+2 > 0)$

iff $\frac{500}{\varepsilon} < n^3 + 2$

iff $\frac{500}{\varepsilon} - 2 < n^3$

iff $n^3 > \frac{500}{\varepsilon} - 2$

iff $n > \left(\frac{500}{\varepsilon} - 2 \right)^{1/3}$.

Now choose integer $N \geq \left(\frac{500}{\varepsilon} - 2\right)^{1/3}$.
Thus, if $n > N$ it follows that
$|a_n - L| < \varepsilon$.

QED

2.) $$\lim_{n\to\infty} \frac{2n+3}{n-10} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2n+3}{n-10} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{2+3/n}{1-10/n} = \frac{2+0}{1-0} = 2.$$

<u>Proof</u>: Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $|a_n - L| < \varepsilon$. Begin with $|a_n - L| < \varepsilon$ and solve for n. Then

$$|a_n - L| < \varepsilon \text{ iff } \left|\frac{2n+3}{n-10} - 2\right| < \varepsilon$$

$$\text{iff } \left|\frac{2n+3-2(n-10)}{n-10}\right| < \varepsilon$$

$$\text{iff } \left|\frac{\cancel{2n}+3-\cancel{2n}+20}{n-10}\right| < \varepsilon$$

$$\text{iff } \frac{23}{|n-10|} < \varepsilon$$

$$\text{iff } \frac{23}{n-10} < \varepsilon$$ (assume $n > 10$ since $n \to \infty$.)

$$\text{iff } \frac{23}{\varepsilon} < n - 10$$

iff $\frac{23}{\varepsilon} + 10 < n$

iff $n > \frac{23}{\varepsilon} + 10$.

Now choose integer $N \geq \frac{23}{\varepsilon} + 10$. Thus, if $n > N$, it follows that $|a_n - L| < \varepsilon$.

3.) $\lim_{n\to\infty} \frac{3n+2}{5-4n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{3n+2}{5-4n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{3+2/n}{5/n - 4} = \frac{3+0}{0-4} = -3/4$.

<u>Proof</u>: Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $|a_n - L| < \varepsilon$. Begin with $|a_n - L| < \varepsilon$ and solve for n. Then

$|a_n - L| < \varepsilon$ iff $\left| \frac{3n+2}{5-4n} - \frac{-3}{4} \right| < \varepsilon$

iff $\left| \frac{4(3n+2) + 3(5-4n)}{4(5-4n)} \right| < \varepsilon$

iff $\left| \frac{\cancel{12n} + 8 + 15 - \cancel{12n}}{4(5-4n)} \right| < \varepsilon$

iff $\frac{23}{|4(5-4n)|} < \varepsilon$

iff $\dfrac{23}{-4(5-4n)} < \varepsilon$ (since $n \to \infty$ we have $5-4n<0$.)

iff $\dfrac{23}{-4\varepsilon} > 5-4n$

iff $4n > 5 + \dfrac{23}{4\varepsilon}$

iff $n > \dfrac{1}{4}\left(5 + \dfrac{23}{4\varepsilon}\right)$.

Now choose integer $N \geq \frac{1}{4}\left(5+\frac{23}{4\varepsilon}\right)$. Thus, if $n > N$, it follows that $|a_n - L| < \varepsilon$.

QED

4.) $\displaystyle \lim_{n\to\infty} \frac{\sqrt{n}+2}{4-\sqrt{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\sqrt{n}+2}{4-\sqrt{n}} \cdot \frac{1/\sqrt{n}}{1/\sqrt{n}}$

$\displaystyle = \lim_{n\to\infty} \frac{1 + 2/\sqrt{n}}{\frac{4}{\sqrt{n}} - 1} = \frac{1+0}{0-1} = -1$.

<u>Proof</u>: Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $|a_n - L| < \varepsilon$. Begin with $|a_n - L| < \varepsilon$ and solve for n. Then

$|a_n - L| < \varepsilon$ iff $\left| \dfrac{\sqrt{n}+2}{4-\sqrt{n}} - -1 \right| < \varepsilon$

iff $\left| \frac{(\cancel{\sqrt{n}}+2)+(4-\cancel{\sqrt{n}})}{4-\sqrt{n}} \right| < \varepsilon$

iff $\frac{6}{|4-\sqrt{n}|} < \varepsilon$

iff $\frac{6}{-(4-\sqrt{n})} < \varepsilon$ (since $n \to \infty$ we know that $4-\sqrt{n}<0$.)

iff $\frac{6}{-\varepsilon} > 4-\sqrt{n}$

iff $\sqrt{n} > 4+\frac{6}{\varepsilon}$

iff $n > \left(4+\frac{6}{\varepsilon}\right)^2$.

Now choose integer $N \geq \left(4+\frac{6}{\varepsilon}\right)^2$. Thus, if $n>N$, it follows that $|a_n-L|<\varepsilon$.

QED

5.) $\lim_{n\to\infty} (0.999)^n = 0$ since $-1<0.999<1$.

<u>Proof</u>: Let $\varepsilon>0$ be given. Find integer N so that if $n>N$ then $|a_n-L|<\varepsilon$. Begin with $|a_n-L|<\varepsilon$ and solve for n. Then

$|a_n-L|<\varepsilon$ iff $|(0.999)^n-0|<\varepsilon$

iff $(0.999)^n < \varepsilon$ (Since $(0.999)^n > 0$)

iff $\ln (0.999)^n < \ln \varepsilon$

iff $n \ln (0.999) < \ln \varepsilon$

iff $n > \dfrac{\ln \varepsilon}{\ln (0.999)}$ (Since $\ln(0.999) < 1$).

Now choose integer $N \geq \dfrac{\ln \varepsilon}{\ln (0.999)}$.

Thus, if $n > N$,
it follows that $|a_n - L| < \varepsilon$.

QED