

# Math 17A

## Kouba

### A Naturally-Occurring Exponential Function

#### Musical Pitch

The pitch of a musical note is determined by the frequency of the vibration which causes it. Middle C on the piano, for example, corresponds to a vibration of 263 hertz (cycles per second). A note one octave above middle C vibrates at 526 hertz, and a note two octaves above middle C vibrates at 1052 hertz. (See Table 1.7.)

TABLE 1.7 Pitch of notes above middle C

Number, $n$ , of octaves above middle C	Number of hertz $V = f(n)$
0	263
1	526
2	1052
3	2104
4	4208

TABLE 1.8 Pitch of notes below middle C

$n$	$V = 263 \cdot 2^n$
-3	$263 \cdot 2^{-3} = 263(1/2^3) = 32.875$
-2	$263 \cdot 2^{-2} = 263(1/2^2) = 65.75$
-1	$263 \cdot 2^{-1} = 263(1/2) = 131.5$
0	$263 \cdot 2^0 = 263$

Notice that

$$\frac{526}{263} = 2 \quad \text{and} \quad \frac{1052}{526} = 2 \quad \text{and} \quad \frac{2104}{1052} = 2$$

and so on. In other words, each value of  $V$  is twice the value before, so

$$\begin{aligned} f(1) &= 526 = 263 \cdot 2 = 263 \cdot 2^1 \\ f(2) &= 1052 = 526 \cdot 2 = 263 \cdot 2^2 \\ f(3) &= 2104 = 1052 \cdot 2 = 263 \cdot 2^3. \end{aligned}$$

In general

$$V = f(n) = 263 \cdot 2^n.$$

The base 2 represents the fact that as we go up an octave, the frequency of vibrations doubles. Indeed, our ears hear a note as one octave higher than another precisely because it vibrates twice as fast. For the negative values of  $n$  in Table 1.8, this function represents the octaves below middle C. The notes on a piano are represented by values of  $n$  between  $-3$  and  $4$ , and the human ear finds values of  $n$  between  $-4$  and  $7$  audible.

Although  $V = f(n) = 263 \cdot 2^n$  makes sense in musical terms only for certain values of  $n$ , values of the function  $f(x) = 263 \cdot 2^x$  can be calculated for all real  $x$ , and its graph has the typical exponential shape, as can be seen in Figure 1.19. It is concave up, climbing faster and faster as  $x$  increases.

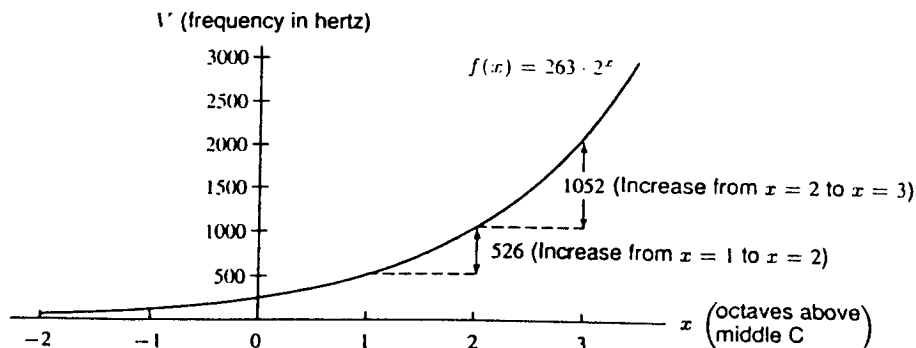


Figure 1.19: Pitch as a function of number of octaves above middle C