Math 17A (Fall 2016)
Kouba
Exam 1

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A GRAPHING CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 8 pages, including the cover page.

6. You may NOT use L'Hopital's Rule on this exam.

7. You may NOT use shortcuts for finding limits to infinity.

8. Using only a calculator to determine limits will receive little or no credit.

9. You will be graded on proper use of limit notation.

10. You have until 11:55 a.m. sharp to finish the exam.

11. STOP IMMEDIATELY WHEN TIME IS CALLED AT THE END AND CLOSE YOUR EXAM. FAILURE TO DO SO MAY LEAD TO A POINTS DEDUCTION ON YOUR EXAM SCORE.

1.) (5 pts. each) Determine the following limits.

a.) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} \frac{(\cancel{x-2})(x+3)}{(\cancel{x-2})(x+2)} = \frac{5}{4}$

b.) $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{1-x} \cdot \frac{\sqrt{x+8}+3}{\sqrt{x+8}+3} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{(x+8)-9}{(1-x)(\sqrt{x+8}+3)}$

$= \lim_{x\to 1} \frac{-(\cancel{1-x})}{(\cancel{1-x})(\sqrt{x+8}+3)} = \frac{-1}{\sqrt{9}+3} = -1/6$

c.) $\lim_{x\to 1} \frac{1-\frac{1}{x}}{1-x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{\frac{x}{x}-\frac{1}{x}}{\frac{1-x}{1}} = \lim_{x\to 1} \frac{x-1}{x} \cdot \frac{1}{1-x}$

$= \lim_{x\to 1} \frac{-(\cancel{1-x})}{x} \cdot \frac{1}{\cancel{1-x}} = \lim_{x\to 1} \frac{-1}{x} = \frac{-1}{1} = -1$

d.) $\lim_{x\to\infty} \frac{4-x^2}{x^2+x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{4-x^2}{x^2+x} \cdot \frac{1/x^2}{1/x^2}$

$= \lim_{x\to\infty} \frac{\cancel{\frac{4}{x^2}}^{\to 0}-1}{1+\cancel{\frac{1}{x}}^{\to 0}} = \frac{0-1}{1+0} = -1$

e.) $\lim_{x\to\infty} \frac{6e^{-2x}+e^{-x}}{2e^{-2x}-e^{-x}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to\infty} \frac{6e^{-2x}+e^{-x}}{2e^{-2x}-e^{-x}} \cdot \frac{e^x}{e^x}$

$= \lim_{x\to\infty} \frac{6e^{-x}+1}{2e^{-x}-1} = \frac{6\cancel{e}^{-\infty}+1}{2\cancel{e}^{-\infty}-1} = \frac{0+1}{0-1} = -1$

2.) a.) (4 pts.) Determine the domain of $f(x) = \dfrac{10}{2 - \sqrt{x-3}}$.

We need $x - 3 \geq 0 \rightarrow x \geq 3$ and $2 - \sqrt{x-3} \neq 0$

$\rightarrow 2 \neq \sqrt{x-3} \rightarrow 4 \neq x - 3 \rightarrow x \neq 7$ so

Domain: all $x \geq 3$ but $x \neq 7$

b.) (4 pts.) Determine the range of $f(x) = 2000 + x^4 + x^2$.

$0 \leq x^4 + x^2 < \infty$ so Range is

all $y \geq 2000$

3.) Assume there are 10,000 birds in a Yolo County wetlands area. Each day 10% of the birds leave to fly south to Mexico and Central America. Let N_t be the number of birds remaining in the wetlands area after t days.

a.) (3 pts.) State the initial value N_0 and give a recursion using N_{t+1} and N_t for $t = 0, 1, 2, 3, \cdots$.

$N_0 = 10{,}000$ birds and

$N_{t+1} = (0.9) N_t$

b.) (5 pts.) Determine N_1 and N_2 and determine an exponential growth formula for N_t for $t = 0, 1, 2, 3, \cdots$.

$N_1 = (0.9) N_0 = (0.9) 10{,}000 = \underline{9000 \text{ birds}}$,

$N_2 = (0.9) N_1 = (0.9)(0.9) N_0 = (0.9)^2 10{,}000 = \underline{8100 \text{ birds}}$,

$\rightarrow \underline{N_t = (0.9)^t N_0 = 10{,}000 (0.9)^t}$

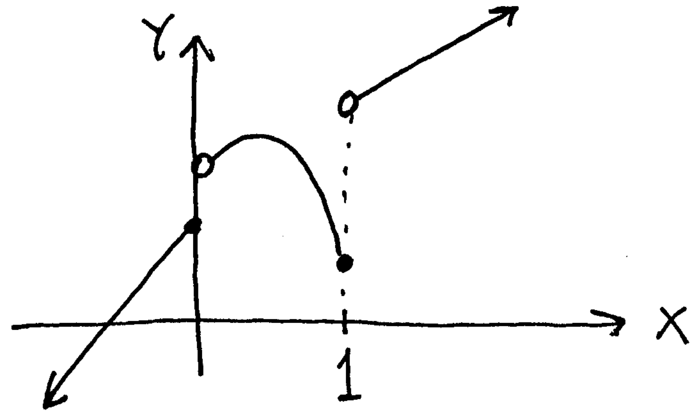
c.) (2 pts.) How many birds remain after 2 weeks ?

2 weeks $\rightarrow t = 14$ days $\rightarrow$

$N_{14} = 10{,}000 (0.9)^{14} \approx 2288$ birds

4.) (7 pts.) Use limits and a "fake graph" to determine the value of constants A and B so that the following function is continuous for all values of x .

$$f(x) = \begin{cases} 4x - A + B, & \text{if } x \leq 0 \\ Bx^2 + Ax + 2\ , & \text{if } 0 < x \leq 1 \\ x + 5, & \text{if } x > 1 \end{cases}$$



We need

$$\lim_{x \to 0^-} (4x - A + B) = \lim_{x \to 0^+} (Bx^2 + Ax + 2)$$

$\rightarrow\ -A + B = 2 \ \rightarrow\ \boxed{B = A + 2}$;

$$\lim_{x \to 1^-} (Bx^2 + Ax + 2) = \lim_{x \to 1^+} (x + 5) \ \rightarrow$$

$B + A + 2 = 6 \ \rightarrow\ \boxed{B + A = 4} \ \rightarrow$ (sub B) $\rightarrow$

$(A + 2) + A = 4 \ \rightarrow\ 2A = 2 \ \rightarrow\ A = 1$, $B = 3$

5.) (5 pts.) Determine all possible fixed points for the following recursion : $a_{n+1} = \dfrac{a_n^2}{a_n^2 - 6}$

$$L = \frac{L^2}{L^2 - 6} \ \rightarrow\ L(L^2 - 6) = L^2$$

$\rightarrow\ L^3 - 6L = L^2$

$\rightarrow\ L^3 - L^2 - 6L = 0$

$\rightarrow\ L(L^2 - L - 6) = 0$

$\rightarrow\ L(L - 3)(L + 2) = 0$

$\rightarrow\ L = 0,\ L = 3,\ L = -2$

6.) (7 pts.) Write an ϵ, N-proof that $\lim_{n\to\infty} \frac{4}{5-\sqrt{n}} = 0$. This is a writing exercise and you will be graded on proper mathematics and style of writing.

Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $\left|\frac{4}{5-\sqrt{n}} - 0\right| < \varepsilon$.
Begin with $\left|\frac{4}{5-\sqrt{n}}\right| < \varepsilon$ and solve for n. Then

$\left|\frac{4}{5-\sqrt{n}}\right| < \varepsilon$

iff $\frac{4}{|5-\sqrt{n}|} < \varepsilon$

(assume $n > 25$ so $5-\sqrt{n} < 0$)

iff $\frac{4}{-(5-\sqrt{n})} < \varepsilon$

iff $\frac{4}{\sqrt{n}-5} < \varepsilon$

iff $\frac{4}{\varepsilon} < \sqrt{n} - 5$

iff $\sqrt{n} > \frac{4}{\varepsilon} + 5$

iff $n > \left(\frac{4}{\varepsilon} + 5\right)^2$. Now choose integer $N \geq \left(\frac{4}{\varepsilon}+5\right)^2$ and the result follows.

Q.E.D.

7.) Find a formula for the nth term (starting with n=0) of each of the following sequences.

a.) (4 pts.) $4, -8, 16, -32, 64, -128, , \cdots$

$2^2, -2^3, 2^4, -2^5, 2^6, \ldots$

$n: 0, 1, 2, 3, 4, \ldots$ $a_n = (-1)^n 2^{n+2}$

b.) (4 pts.) $\frac{1}{4}, \frac{2}{7}, \frac{3}{10}, \frac{4}{13}, \frac{5}{16}, \frac{6}{19}, \cdots$

$n: 0, 1, 2, 3, 4, 5, \ldots$ $a_n = \frac{n+1}{4+3n}$

c.) (6 pts.) $3, 6, 10, 15, 21, 28, 36, \cdots$

n	a_n
0	$3 = 1+2$
1	$6 = 1+2+3$
2	$10 = 1+2+3+4$
3	$15 = 1+2+3+4+5$
4	$21 = 1+2+3+4+5+6$
$\vdots$	$\vdots$
n	$1+2+3+\cdots+(n+2) \rightarrow$

$$a_n = \frac{1}{2}(n+2)\left((n+2)+1\right)$$

8.) (7 pts.) Use the Squeeze Principle (Sandwich Theorem) to determine the limit of the following sequence :

$a_n = \dfrac{5 - 3\cos 2n}{n + 10}$; $\quad -1 \le \cos 2n \le +1 \longrightarrow$

$$+3 \ge -3\cos 2n \ge -3 \longrightarrow$$

$$8 \ge 5 - 3\cos 2n \ge 2 \longrightarrow$$

$$\frac{8}{n+10} \ge \frac{5 - 3\cos 2n}{n+10} \ge \frac{2}{n+10} \; ;$$

$\lim\limits_{n\to\infty} \dfrac{8}{n+10} = 0 = \lim\limits_{n\to\infty} \dfrac{2}{n+10}$, so by Squeeze Principle

$$\lim_{n\to\infty} \frac{5 - 3\cos 2n}{n+10} = 0.$$

9.) Consider the Beverton-Holt Recursion given by $N_{t+1} = \dfrac{100N_t}{5+N_t}$ with initial amount $N_0 = 15$.

a.) (4 pts.) Determine the carrying capacity K.

$$L = \frac{100L}{5+L} \rightarrow 5L + L^2 = 100L \rightarrow$$

$$L^2 - 9L = L(L-95) = 0 \rightarrow L = 0,\ L = 95$$

$$\rightarrow \boxed{K = 95}$$

b.) (4 pts.) Determine the growth parameter R.

$$N_{t+1} = \frac{100\,N_t}{5+N_t} \cdot \frac{1/5}{1/5} = \frac{20\,N_t}{1+\frac{1}{5}N_t} \rightarrow$$

$$\boxed{R = 20}$$

c.) (2 pts.) Find N_1. $N_0 = 15 \rightarrow$

$$N_1 = \frac{100\,N_0}{5+N_0} = \frac{100(15)}{5+(15)} = \frac{150\cancel{0}}{2\cancel{0}} = 75$$

10.) (7 pts.) Show algebraically that the function $f(x) = \dfrac{1-x}{x+2}$ is one-to-one.

Let $f(x_1) = f(x_2) \rightarrow$

$$\frac{1-x_1}{x_1+2} = \frac{1-x_2}{x_2+2} \rightarrow$$

$$(1-x_1)(x_2+2) = (1-x_2)(x_1+2) \rightarrow$$

$$x_2 - \cancel{x_1x_2} + \cancel{2} - 2x_1 = x_1 - \cancel{x_1x_2} + \cancel{2} - 2x_2 \rightarrow$$

$$3x_2 = 3x_1 \rightarrow$$

$x_2 = x_1$ and f is 1-1

The following EXTRA CREDIT PROBLEM is worth 8 points. This problem is OPTIONAL.

1.) Determine the hidden pattern in the following sequence and then find the next two terms in the sequence.

2, 6, 13, 24, 40, 62, 91, 128, ...

2, 6, 13, 24, 40, 62, 91, 128, $\boxed{174}$, $\boxed{230}$

4 7 11 16 22 29 37 46 56

3 4 5 6 7 8 9 10