Math 17A (Fall 2016)
Kouba
Exam 3

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 8 pages, including the cover page.

6. Put units on answers where units are appropriate.

7. You will be graded on proper use of derivative notation.

8. You have until 11:58 a.m. sharp to finish the exam. Failure to stop working and close your exam when time is called may lead to a points deduction to your exam score.

1.) (6 pts. each) Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

a.) $y = \ln(2x+3) + \log_5 x$

$$\xrightarrow{D} y' = \frac{1}{2x+3}\cdot 2 + \frac{1}{x}\cdot\frac{1}{\ln 5}$$

b.) $f(x) = \sin^2(7^x)$

$$\xrightarrow{D} f'(x) = 2\sin(7^x)\cdot\cos(7^x)\cdot 7^x \ln 7$$

c.) $f(x) = (\sin x)^{x+1}$ $\rightarrow$ $\ln f(x) = \ln(\sin x)^{x+1} \rightarrow$

$$\ln f(x) = (x+1)\ln(\sin x) \xrightarrow{D}$$

$$\frac{1}{f(x)} f'(x) = (x+1)\cdot\frac{\cos x}{\sin x} + (1)\ln(\sin x) \rightarrow$$

$$f'(x) = (\sin x)^{x+1}\left[(x+1)\frac{\cos x}{\sin x} + \ln(\sin x)\right]$$

d.) $f(x) = x^x \cdot e^x \cdot \sin x \cdot \cos x$

$$\ln f(x) = x\ln x + x + \ln(\sin x) + \ln(\cos x) \xrightarrow{D}$$

$$\frac{1}{f(x)} f'(x) = x\cdot\frac{1}{x} + (1)\ln x + 1 + \frac{\cos x}{\sin x} + \frac{-\sin x}{\cos x} \rightarrow$$

$$f'(x) = x^x e^x \sin x \cos x\left[2 + \ln x + \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}\right]$$

2.) (7 pts.) Determine the x-values for which the function $f(x) = x(x-6)^5$ is increasing (↑) and decreasing (↓). DO NOT GRAPH THE FUNCTION.

$$\xrightarrow{D} f'(x) = x\cdot 5(x-6)^4 + (1)(x-6)^5$$

$$= (x-6)^4[5x + (x-6)]$$

$$= (x-6)^4[6x-6] = 0$$

$-$ 0 $+$ 0 $+$ f'

$x=1$ $x=6$

f is ↑ for $1 < x < 6$, $x > 6$,

f is ↓ for $x < 1$.

3.) (12 pts.) For the following function f determine its domain and all absolute and relative maximum and minimum values, inflection points, and x- and y-intercepts. State clearly the x-values for which f is increasing ($\uparrow$), decreasing ($\downarrow$), concave up ($\bigcup$), and concave down ($\bigcap$). Neatly sketch the graph of f. The fucntion and its derivatives are given below:

$$f(x) = x - 6 \cdot \sqrt{x} \;,\; f'(x) = \frac{\sqrt{x}-3}{\sqrt{x}} \;,\; f''(x) = \frac{3}{2x^{3/2}}$$

Domain : all $x \geq 0$

$f'(x) = \frac{\sqrt{x}-3}{\sqrt{x}} = 0$

$\rightarrow \sqrt{x} - 3 = 0 \rightarrow$

$x = 9$ and $x \neq 0$

NO | $-$ | 0 | $+$ | f'

rel. max. $\left\{ \begin{array}{l} X=0 \\ Y=0 \end{array} \right.$ $\left. \begin{array}{l} x=9 \\ Y=-9 \end{array} \right\}$ abs. min.

$f''(x) = \frac{3}{2x^{3/2}} \neq 0$

NO | $+$ | f''

$X=0$

$X=0 : Y=0$

$Y=0 : \sqrt{x}(\sqrt{x}-6)=0$

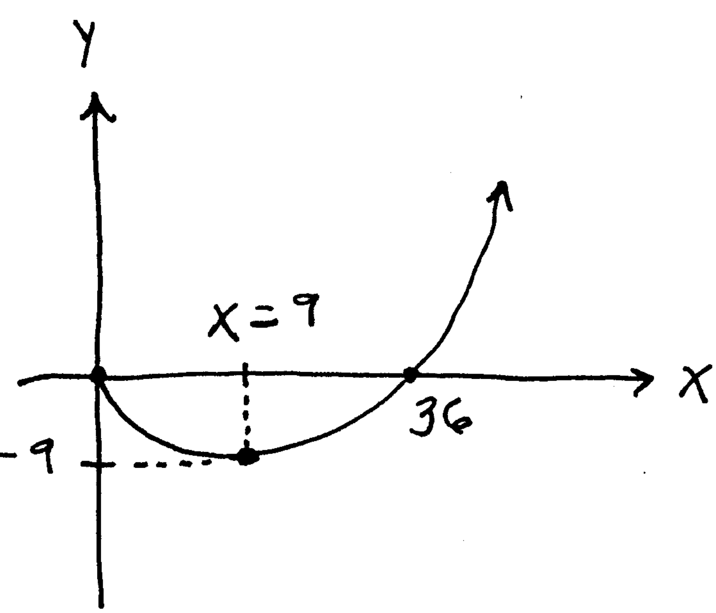
$\rightarrow X=0, X=36$

f is $\uparrow$ for $x > 9$,

f is $\downarrow$ for $0 < x < 9$

$\lim_{x\to\infty} \sqrt{x}(\sqrt{x}-6) = \infty$

so $(0,0)$ is rel. max.



f is $\bigcup$ for $X > 0$

4.) (7 pts.) Cesium-137 is an isotope produced by nuclear fission, and is used in medical radiation therapy devices for treating cancer. It's half-life is 30.17 years. If a sample of Cs-137 has an initial mass of 100 mg, how much will remain after 250 years ?

$$N = ce^{kt} \rightarrow t=0,\ N=100 \rightarrow N = 100e^{kt};$$

$$t = 30.17,\ N = 50 \rightarrow 50 = 100e^{k(30.17)} \rightarrow$$

$$\frac{1}{2} = e^{30.17k} \rightarrow \ln(1/2) = \ln e^{30.17k} = 30.17k \rightarrow$$

$$k = \frac{1}{30.17}\ln(1/2) \rightarrow N = 100e^{\frac{\ln(1/2)}{30.17}t};$$

let $t = 250 \rightarrow$

$$N = 100e^{\frac{\ln(1/2)\cdot 250}{30.17}} \approx \mathbf{0.32\ mg}$$

5.) (7 pts.) Consider the function $f(x) = x + \ln x$ defined on the closed interval $[1, e]$. Verify that f satisfies the assumptions of the Mean Value Theorem (MVT) and find all values of c guaranteed by the MVT.

$f(x) = x + \ln x$ is cont. on $[1, e]$ (sum of cont. fcns.); $f'(x) = 1 + \frac{1}{x}$, so f is diff. on $(1, e)$. Thus, by MVT there is at least one # c, $1 < c < e$, satisfying

$$f'(c) = \frac{f(e) - f(1)}{e-1} = \frac{(e + \cancel{\ln e}^{1}) - (1 + \cancel{\ln 1}^{0})}{e-1} = \frac{e}{e-1} \rightarrow$$

$$1 + \frac{1}{c} = \frac{e}{e-1} \rightarrow \frac{1}{c} = \frac{e}{e-1} - \frac{e-1}{e-1} = \frac{1}{e-1} \rightarrow$$

$$c = e-1$$

6.) (7 pts.) Consider the function $f(x) = x + x^3$. Use a derivative to verify that f is one-to-one (and therefore has an inverse function f^{-1}) . If $f(2) = 10$, then what is the value of $Df^{-1}(10)$?

$\xrightarrow{D}$ $f'(x) = 1 + 3x^2 > 0$ so f is $\uparrow$ and f is 1-1;

$$Df^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \rightarrow$$

$$Df^{-1}(10) = \frac{1}{f'(f^{-1}(10))} = \frac{1}{f'(2)} = \frac{1}{1+3(2)^2} = \frac{1}{13}$$

7.) (7 pts.) Use a linearization to estimate the value of $(18)^{1/4}$.

Let $f(x) = x^{1/4}$ and choose $a = 16$;

$\xrightarrow{D}$ $f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$; lin. is

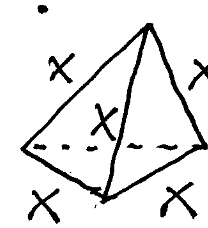
$$L(x) = f(16) + f'(16)(x-16)$$
$$= 16^{1/4} + \frac{1}{4 \cdot 16^{3/4}}(x-16) = 2 + \frac{1}{32}(x-16) \rightarrow$$

$L(x) = \frac{3}{2} + \frac{1}{32}x$; then $f(18) \approx L(18) \rightarrow$

$$(18)^{1/4} \approx \frac{3}{2} + \frac{1}{32}(18) = \frac{24}{16} + \frac{9}{16} = \frac{33}{16} = 2.0625$$

8.) (7 pts.) The edge x of a regular tetrahedron (pyramid) is measured with an absolute percentage error of at most 2 percent. Use a differential to estimate the maximum absolute percentage error in computing the tetrahedron's volume. You will be graded on proper use of notation. HINT: The volume of the tetrahedron is $V = \frac{x^3}{6\sqrt{2}}$.

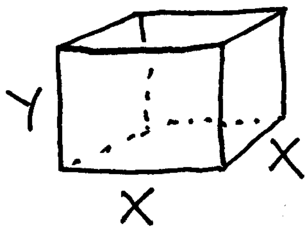
Given $\frac{|\Delta x|}{x} \leq 2\%$; find $\frac{|\Delta V|}{V}$:



$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta x|}{V}$$

$$= \frac{\left|\cancel{\frac{1}{6\sqrt{2}}} \cdot 3x^2 \cdot \Delta x\right|}{\cancel{\frac{1}{6\sqrt{2}}}\, x^3} = 3\frac{|\Delta x|}{x} \leq 3(2\%) = 6\%$$

9.) (7 pts.) An open rectangular box with a square base is to be made from 150 cm.^{2} of material. What dimensions will result in the box of largest possible volume and what is the largest volume ?



Given $X^2 + 4XY = 150 \rightarrow$

$4XY = 150 - X^2 \rightarrow \boxed{Y = \frac{150 - X^2}{4X}}$;

max. volume

$$V = x^2 Y = x^2\left(\frac{150 - X^2}{4X}\right) = \frac{1}{4}(150X - x^3) \xrightarrow{D}$$

$$V' = \frac{1}{4}(150 - 3X^2) = \frac{3}{4}(50 - X^2) = 0$$

$+ \quad 0 \quad -$ V'

$X = \sqrt{50} = 5\sqrt{2}$ cm.

$Y = \frac{100}{20\sqrt{2}} = \frac{5}{\sqrt{2}}$ cm.

and Max. $V = \frac{250}{\sqrt{2}}$ cm.^{3}

10.) (7 pts.) The Bedbug Motel fills 100 rooms if the charge per room is \$40, for a daily revenue of (100)(40) = \$4000. For each \$5 increase in price, 4 fewer rooms are filled. Use calculus to determine what charge per room (and number of rooms) will maximize revenue. What is the maximum revenue ?

Let X : # of \$5 increases; maximize revenue R = (charge per room)(# of rooms) $\rightarrow$

$$R = (40 + 5X)(100 - 4X) \xrightarrow{D}$$

$$R' = (40 + 5X)(-4) + (5)(100 - 4X)$$

$$= -160 - 20X + 500 - 20X$$

$$= 340 - 40X = 20(17 - 2X) = 0 \rightarrow X = 8.5$$

$+ \quad 0 \quad -$ R'

$X = 8.5$ $\uparrow$'s

rooms = 66

\$ / room = \$82.50

MAX revenue $R = \$5445$

11.) (8 pts.) Assume that the derivative of function $y = f(x)$ is $f'(x) = \frac{x}{x^2+9}$. Find all x-values corresponding to inflection points for the function f.

$$\xrightarrow{D} f''(x) = \frac{(x^2+9)(1) - x(2x)}{(x^2+9)^2}$$

$$= \frac{x^2+9-2x^2}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2} = \frac{(3-x)(3+x)}{(x^2+9)^2} = 0;$$

$$-\quad 0 \quad + \quad 0 \quad - \qquad f''$$

$$x=-3 \qquad x=3$$

$$\uparrow \qquad \uparrow$$

Both are Infl. Pts.

The following two EXTRA CREDIT PROBLEM are worth 6 points each. These problems are OPTIONAL.

1.) Show that the function $y = \frac{e^x - 1}{2+e^x}$ is one-to-one and find its inverse function.

$$\xrightarrow{D} y' = \frac{(2+e^x)\cdot e^x - (e^x-1)\cdot e^x}{(2+e^x)^2}$$

$$= \frac{2e^x + \cancel{e^{2x}} - \cancel{e^{2x}} + e^x}{(2+e^x)^2} = \frac{3e^x}{(2+e^x)^2} > 0$$

so Y is $\uparrow$ and Y is 1-1 ; Find $f^{-1}(x)$:

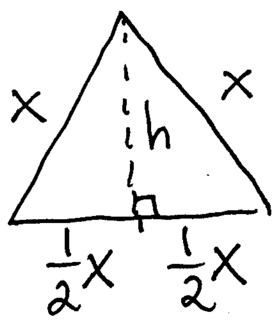
$$x = \frac{e^Y-1}{2+e^Y} \rightarrow x(2+e^Y) = e^Y - 1 \rightarrow$$

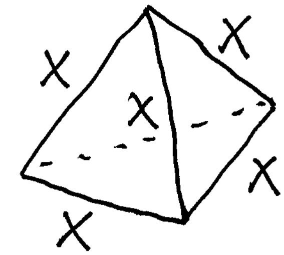
$$2x + xe^Y = e^Y - 1 \rightarrow xe^Y - e^Y = -2x-1 \rightarrow$$

$$(x-1)e^Y = -2x-1 \rightarrow e^Y = \frac{-2x-1}{x-1} = \frac{2x+1}{1-x} \rightarrow$$

$$Y = \ln\left(\frac{2x+1}{1-x}\right) = f^{-1}(x)$$

2.) Use a differential to estimate the absolute percentage error in computing the tetrahedron's surface area in Problem 8.





$h^2 + \left(\frac{1}{2}x\right)^2 = x^2 \to h^2 = \frac{3}{4}x^2 \to$

$h = \frac{\sqrt{3}}{2}x$ so area of $\triangle$ is

$A = \frac{1}{2}(x) \cdot \frac{\sqrt{3}}{2}x = \frac{\sqrt{3}}{4}x^2 \to$

surface area is $A = 4\left(\frac{\sqrt{3}}{4}x^2\right) \to$

$\boxed{A = \sqrt{3}x^2}$; given $\frac{|\Delta x|}{x} \le 2\%$,

find $\frac{|\Delta A|}{A}$:

$$\frac{|\Delta A|}{A} \approx \frac{|dA|}{A} = \frac{|A' \cdot \Delta x|}{A} = \frac{|2\cancel{\sqrt{3}}x \cdot \Delta x|}{\cancel{\sqrt{3}}x^2}$$

$$= 2\frac{|\Delta x|}{x} \le 2(2\%) = 4\%$$