Math 17B

Kouba

Laplace Transforms – An Example Using Integration by Parts Twice

Ex: $f(t) = \sin t \longrightarrow \mathcal{L}\{\sin t\} = \int_0^\infty e^{-st} \cdot \sin t \, dt$

$$= \lim_{A \to \infty} \underbrace{\int_0^A e^{-st} \sin t \, dt}_{\text{call it } K} \quad (\text{assume } s > 0.)$$

call it K : (Let $u = e^{-st}$, $dv = \sin t \, dt$

$\longrightarrow$ $du = -se^{-st} dt$, $v = -\cos t$)

$$K = \int_0^A e^{-st} \sin t \, dt = -\cos t \, e^{-st} \Big|_0^A - s \int_0^A e^{-st} \cos t \, dt$$

$$= \left(\frac{-\cos A}{e^{sA}} - -1 \right) - s \int_0^A e^{-st} \cos t \, dt$$

(Let $u = e^{-st}$, $dv = \cos t \, dt$

$\longrightarrow$ $du = -se^{-st}$, $v = \sin t$)

$$= \left(-\frac{\cos A}{e^{sA}} + 1 \right) - s \left[e^{-st} \sin t \Big|_0^A - -s \int_0^A e^{-st} \sin t \, dt \right]$$

$$= \frac{-\cos A}{e^{sA}} + 1 - s \left[\frac{\sin A}{e^{sA}} - 0 + s \cdot \underbrace{\int_0^A e^{-st} \sin t \, dt}_{K} \right] \longrightarrow$$

$$K = \frac{-\cos A}{e^{sA}} + 1 - s \cdot \frac{\sin A}{e^{sA}} - s^2 K \longrightarrow$$

$$K + s^2 K = \frac{-\cos A}{e^{sA}} + 1 - s \cdot \frac{\sin A}{e^{sA}} \longrightarrow$$

$$(1+s^2)K = \frac{-\cos A}{e^{sA}} + 1 - s \cdot \frac{\sin A}{e^{sA}} \longrightarrow$$

$$K = \frac{1}{1+s^2}\left[\frac{-\cos A}{e^{sA}} + 1 - s \cdot \frac{\sin A}{e^{sA}}\right];$$

then

By Squeeze Principle

$$\mathcal{L}\{\sin t\} = \lim_{A\to\infty} K$$

$$= \lim_{A\to\infty} \frac{1}{1+s^2}\left[\cancel{\frac{-\cos A}{e^{sA}}}^{\,0} + 1 - s \cdot \cancel{\frac{\sin A}{e^{sA}}}^{\,0}\right]$$

$$= \frac{1}{1+s^2}[1] \longrightarrow \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

for $s > 0$.