

1.) Solve the following first-order linear differential equations.

a.) $\frac{dy}{dx} + 2y = 5$

b.) $\frac{dy}{dx} + y = e^{3x}$

c.) $y' + 3x^2y = x^2$

d.) $x^2y' + xy = 1$

e.) $(1 + x^2)y' + xy + x^3 + x = 0$

f.) $xy' + (1 + x)y = e^{-x} \sin 2x$

g.) $\frac{dy}{dx} = y + x$

h.) $y' = 2y + xe^{2x}$ and $y(0) = 2$

i.) $\cos x \cdot \frac{dy}{dx} + y \sin x = 1$

j.) $y' + y = \frac{1 - e^{-2x}}{e^x + e^{-x}}$

k.) $(1 + x)y' - xy = x + x^2$

l.) $\cos^2 x \sin x \cdot \frac{dy}{dx} + y \cos^3 x = 1$

First-Order Linear Solutions

$$1.) a.) y' + 2y = 5 \quad \{ \mu = e^{\int 2 dx} = e^{2x} \} \rightarrow$$

$$e^{2x} y' + 2e^{2x} y = 5e^{2x} \rightarrow D[e^{2x} y] = 5e^{2x} \rightarrow$$

$$e^{2x} y = \int 5e^{2x} dx \rightarrow e^{2x} y = 5 \cdot \frac{1}{2} e^{2x} + c \rightarrow$$

$$\boxed{e^{2x} y = \frac{5}{2} e^{2x} + c}$$

$$b.) y' + y = e^{3x} \quad \{ \mu = e^{\int 1 dx} = e^x \} \rightarrow$$

$$e^x y' + e^x y = e^x \cdot e^{3x} = e^{4x} \rightarrow$$

$$D[e^x y] = e^{4x} \rightarrow e^x y = \int e^{4x} dx \rightarrow$$

$$\boxed{e^x y = \frac{1}{4} e^{4x} + c}$$

$$c.) y' + 3x^2 y = x^2 \quad \{ \mu = e^{\int 3x^2 dx} = e^{x^3} \} \rightarrow$$

$$e^{x^3} y' + 3x^2 \cdot e^{x^3} \cdot y = x^2 e^{x^3} \rightarrow$$

$$D[e^{x^3} y] = x^2 e^{x^3} \rightarrow e^{x^3} y = \int x^2 e^{x^3} dx \rightarrow$$

$$\boxed{e^{x^3} y = \frac{1}{3} e^{x^3} + c}$$

$$d.) x^2 y' + xy = 1 \rightarrow y' + \frac{1}{x} \cdot y = \frac{1}{x^2}$$

$$\{ \mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x \} \rightarrow$$

$$xy' + y = \frac{1}{x} \rightarrow D[xy] = \frac{1}{x} \rightarrow$$

$$xy = \int \frac{1}{x} dx \rightarrow \boxed{xy = \ln|x| + C}$$

$$e.) (1+x^2)y' + xy + x^3 + x = 0 \rightarrow$$

$$(1+x^2)y' + xy = -x^3 - x \rightarrow$$

$$y' + \frac{x}{1+x^2} y = \frac{-x(x^2+1)}{1+x^2} = -x$$

$$\left\{ \begin{aligned} \mu &= e^{\int \frac{x}{1+x^2} dx} = e^{\frac{1}{2} \ln(1+x^2)} = e^{\ln(1+x^2)^{\frac{1}{2}}} \\ &= \sqrt{1+x^2} \end{aligned} \right\} \rightarrow$$

$$(1+x^2)^{\frac{1}{2}} y' + \frac{x}{(1+x^2)^{\frac{1}{2}}} y = -x(1+x^2)^{\frac{1}{2}} \rightarrow$$

$$D[(1+x^2)^{\frac{1}{2}} y] = -x(1+x^2)^{\frac{1}{2}} \rightarrow$$

$$(1+x^2)^{\frac{1}{2}} y = -\int x(1+x^2)^{\frac{1}{2}} dx \rightarrow$$

$$\boxed{(1+x^2)^{\frac{1}{2}} y = -\frac{2}{3} \cdot \frac{1}{2} (1+x^2)^{\frac{3}{2}} + C}$$

$$f.) xy' + (1+x)y = e^{-x} \sin 2x \rightarrow$$

$$y' + \left(\frac{1}{x} + 1\right)y = \frac{1}{x} e^{-x} \sin 2x$$

$$\left\{ \begin{aligned} \mu &= e^{\int \left(\frac{1}{x} + 1\right) dx} = e^{\ln x + x} \\ &= e^{\ln x} \cdot e^x = x e^x \end{aligned} \right\} \rightarrow$$

$$xe^x y' + (1+x)e^x y = \sin 2x \rightarrow$$

$$D[xe^x y] = \sin 2x \rightarrow xe^x y = \int \sin 2x dx$$

$$\rightarrow \boxed{xe^x y = -\frac{1}{2} \cos 2x + c}$$

$$g.) y' - y = x \quad \left\{ \mu = e^{\int -1 dx} = e^{-x} \right\} \rightarrow$$

$$e^{-x} y' - e^{-x} y = x e^{-x} \rightarrow D[e^{-x} y] = x e^{-x} \rightarrow$$

$$e^{-x} y = \int x e^{-x} dx \quad (\text{let } u = x, dv = e^{-x} dx$$

$$\rightarrow du = 1 dx, v = -e^{-x})$$

$$= -x e^{-x} - \int e^{-x} dx$$

$$= -x e^{-x} + -e^{-x} + c \rightarrow$$

$$\boxed{e^{-x} y = -x e^{-x} - e^{-x} + c}$$

$$h.) y' - 2y = x e^{2x} \quad \left\{ \mu = e^{\int -2 dx} = e^{-2x} \right\} \rightarrow$$

$$e^{-2x} y' - 2e^{-2x} y = x \rightarrow$$

$$D[e^{-2x} y] = x \rightarrow e^{-2x} y = \int x dx \rightarrow$$

$$e^{-2x} y = \frac{1}{2} x^2 + c \quad \text{and } x=0, y=2 \rightarrow$$

$$1)(2) = 0 + c \rightarrow c = 2 \rightarrow \boxed{e^{-2x} y = \frac{1}{2} x^2 + 2}$$

$$i.) \cos x \cdot y' + \sin x \cdot y = 1 \rightarrow$$

$$y' + \frac{\sin x}{\cos x} y = \frac{1}{\cos x}$$

$$\left\{ \mu = e^{\int \frac{\sin x}{\cos x} dx} = e^{-\ln(\cos x)} \right.$$

$$\left. = e^{\ln(\cos x)^{-1}} = \frac{1}{\cos x} \right\} \rightarrow$$

$$\frac{1}{\cos x} \cdot y' + \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot y = \frac{1}{\cos^2 x} \rightarrow$$

$$\sec x \cdot y' + \sec x \cdot \tan x \cdot y = \sec^2 x \rightarrow$$

$$D[\sec x \cdot y] = \sec^2 x \rightarrow$$

$$\sec x \cdot y = \int \sec^2 x dx \rightarrow$$

$$\boxed{\sec x \cdot y = \tan x + c}$$

$$j.) y' + y = \frac{1 - e^{-2x}}{e^x + e^{-x}} \left\{ \mu = e^{\int 1 dx} = e^x \right\} \rightarrow$$

$$e^x y' + e^x y = \frac{e^x - e^{-x}}{e^x + e^{-x}} \rightarrow$$

$$D[e^x y] = \frac{e^x - e^{-x}}{e^x + e^{-x}} \rightarrow$$

$$e^x y = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx \rightarrow$$

$$e^x y = \ln(e^x + e^{-x}) + c$$

$$k.) (1+x)y' - xy = x + x^2 = x(1+x) \rightarrow$$

$$y' + \frac{-x}{1+x} \cdot y = \frac{x(1+x)}{(1+x)} = x$$

$$\left\{ \begin{aligned} \mu &= e^{\int \frac{-x}{1+x} dx} : \text{let } u=1+x, x=u-1 \text{ and } du=dx, \\ &= e^{\int -\frac{(u-1)}{u} du} = e^{\int (-1 + \frac{1}{u}) du} \\ &= e^{-u + \ln u} = e^{-(1+x) + \ln(1+x)} = \frac{1+x}{e^{1+x}} \end{aligned} \right\} \rightarrow$$

$$\frac{1+x}{e^{1+x}} \cdot y' - \frac{x}{e^{1+x}} \cdot y = \frac{x+x^2}{e^{1+x}} \rightarrow$$

$$D \left[\frac{1+x}{e^{1+x}} \cdot y \right] = (x+x^2) e^{-1-x} \rightarrow$$

$$\frac{1+x}{e^{1+x}} \cdot y = \int (x+x^2) e^{-1-x} dx$$

$$\left\{ \begin{aligned} \text{Let } u &= x+x^2, \quad dv = e^{-1-x} dx \\ \rightarrow du &= (1+2x) dx, \quad v = -e^{-1-x} \end{aligned} \right\}$$

$$= -(x+x^2) e^{-1-x} - \int (1+2x) e^{-1-x} dx$$

$$\left\{ \begin{aligned} \text{Let } u &= 1+2x, \quad dv = e^{-1-x} dx \\ \rightarrow du &= 2 dx, \quad v = -e^{-1-x} \end{aligned} \right\}$$

$$\begin{aligned}
&= -(x+x^2)e^{-1-x} \\
&\quad + \left[-(1+2x)e^{-1-x} - 2 \int e^{-1-x} dx \right] \\
&= -xe^{-1-x} - x^2e^{-1-x} - e^{-1-x} - 2xe^{-1-x} \\
&\quad + 2 \cdot (-1)e^{-1-x} + c \\
&= -3e^{-1-x} - 3xe^{-1-x} - x^2e^{-1-x} + c \rightarrow
\end{aligned}$$

$$\boxed{\frac{1+x}{e^{1+x}} \cdot y = -3e^{-1-x} - 3xe^{-1-x} - x^2e^{-1-x}}$$

$$1.) \cos^2 x \cdot \sin x \cdot y' + \cos^3 x \cdot y = 1 \rightarrow$$

$$y' + \frac{\cos x}{\sin x} \cdot y = \frac{1}{\cos^2 x \cdot \sin x}$$

$$\left\{ \mu = e^{\int \frac{\cos x}{\sin x} dx} = e^{\ln(\sin x)} = \sin x \right\} \rightarrow$$

$$\sin x \cdot y' + \cos x \cdot y = \frac{1}{\cos^2 x} = \sec^2 x \rightarrow$$

$$D[\sin x \cdot y] = \sec^2 x \rightarrow \sin x \cdot y = \int \sec^2 x dx$$

$$\rightarrow \boxed{\sin x \cdot y = \tan x + c}$$

Math 17B
Kouba
Worksheet 2

Let S represent the amount (in pounds) of salt in each tank at time t minutes. Find a formula for S for each of the following and then answer the particular questions.

1.) A solution containing $1/2$ lb. of salt per gallon flows into a tank at the rate of 2 gal./min. and the well-stirred mixture flows out of the tank at the same rate. The tank initially holds 100 gallons of solution containing 5 lbs. of salt.

- a.) How much salt is in the tank after 10 minutes ? after 1 hour ?
- b.) How much salt do you expect to be in the tank as t gets infinitely large ?

2.) Pure water flows into a tank at the rate of 4 gal./min. and the well-stirred mixture flows out of the tank at the rate of 5 gal./min. The tank initially holds 200 gallons of water containing 50 lbs. of salt.

- a.) How many gallons of solution are in the tank after 20 minutes ?
- b.) How much salt is in the tank after 20 minutes ? after 2 hours ?
- c.) How long will it take the tank to be empty ?

3.) A large tank holds 100 gallons of fluid in which 10 pounds of salt is dissolved. A mixture containing $1/2$ lb. of salt per gallon flows into the tank at the rate of 6 gal./min. and the well-stirred mixture flows out of the tank at the rate of 4 gal./min.

- a.) How many gallons of solution are in the tank after 10 minutes ? after 1 hour ?
- b.) How much salt is in the tank after 10 minutes ? after 1 hour ?
- c.) In how many minutes will the tank contain 40 pounds of salt ? (HINT: Use a calculator with an equation solver to solve for t or just estimate the solution by trial-and-error.)

4.) Beer containing 6% alcohol per gallon is pumped into a vat which initially contains 400 gallons of beer at 3% alcohol. Beer is pumped into the tank at

the rate of 3 gal./min. and the well-stirred mixture is pumped out of the tank at the rate of 4 gal./min.

a.) How many gallons of beer are in the vat after 10 minutes ? after 1 hour ? after 6 hours and 40 minutes ?

b.) What is the percentage of alcohol in the vat after 10 minutes ? after 1 hour ?

c.) When will the percentage of alcohol in the vat be 4% ? (HINT: Use a calculator with an equation solver to solve for t or just estimate the solution by trial-and-error.)

Worksheet 2

Solutions

Let S be lbs. of salt in tank at time t .

$$1.) \frac{dS}{dt} = (\text{Rate In}) - (\text{Rate Out})$$

$$= \left(\frac{\frac{1}{2} \text{ lb.}}{\text{gal.}}\right) \left(\frac{2 \text{ gal.}}{\text{min.}}\right) - \left(\frac{S \text{ lbs.}}{100 \text{ gal.}}\right) \left(\frac{2 \text{ gal.}}{\text{min.}}\right) \rightarrow$$

$$\boxed{\frac{dS}{dt} = 1 - \frac{1}{50} S} \rightarrow \int \frac{1}{1 - \frac{1}{50} S} dS = \int dt \rightarrow$$

$$-50 \ln \left| 1 - \frac{1}{50} S \right| = t + C \quad \text{and}$$

$$t = 0 \text{ min.}, \quad S = 5 \text{ lbs.} \rightarrow$$

$$-50 \ln \left(\frac{9}{10}\right) = 0 + C \rightarrow C = -50 \ln \left(\frac{9}{10}\right) \rightarrow$$

$$-50 \ln \left(1 - \frac{1}{50} S\right) = t - 50 \ln \left(\frac{9}{10}\right) \rightarrow$$

(since $S < 50$) \rightarrow

$$\ln \left(1 - \frac{1}{50} S\right) = \frac{-1}{50} t + \ln \left(\frac{9}{10}\right) \rightarrow$$

$$e^{\ln \left(1 - \frac{1}{50} S\right)} = e^{\frac{-1}{50} t + \ln \left(\frac{9}{10}\right)} \rightarrow$$

$$1 - \frac{1}{50} S = e^{\frac{-1}{50} t} e^{\ln \left(\frac{9}{10}\right)} \rightarrow$$

$$1 - \frac{1}{50} S = \frac{9}{10} e^{\frac{-1}{50} t} \rightarrow$$

$$\frac{1}{50} S = 1 - \frac{9}{10} e^{\frac{-1}{50} t} \rightarrow$$

$$\boxed{S = 50 - 45 e^{\frac{-1}{50} t}}$$

$$a.) \quad t = 10 \text{ min.} \rightarrow S = 50 - 45e^{-\frac{1}{5}t} \approx 13.2 \text{ lbs.}$$

$$t = 60 \text{ min.} \rightarrow S = 50 - 45e^{-\frac{6}{5}} \approx 36.4 \text{ lbs.}$$

$$b.) \quad \lim_{t \rightarrow \infty} S = \lim_{t \rightarrow \infty} (50 - 45e^{-\frac{1}{50}t})$$

$$= 50 - 45(e^{-\infty}) = 50 - 45(0)$$

$$= 50 \text{ lbs.}$$

Let S be lbs. of salt in tank at time t

$$2.) \quad \frac{dS}{dt} = (\text{Rate In}) - (\text{Rate Out})$$

$$= \left(\frac{0 \text{ lbs.}}{\text{gal.}} \right) \left(\frac{4 \text{ gal.}}{\text{min.}} \right) - \left(\frac{S \text{ lbs.}}{200-t \text{ gal.}} \right) \left(\frac{5 \text{ gal.}}{\text{min.}} \right) \rightarrow$$

$$\boxed{\frac{dS}{dt} = \frac{-5S}{200-t}} \rightarrow \int \frac{1}{S} dS = \int \frac{-5}{200-t} dt \rightarrow$$

$$\ln S = 5 \ln(200-t) + c ; \text{ and}$$

$$t = 0 \text{ min.}, S = 50 \text{ lbs.} \rightarrow$$

$$\ln 50 = 5 \ln 200 + c \rightarrow$$

$$c = \ln 50 - \ln 200^5 = \ln \left(\frac{50}{200^5} \right) \rightarrow$$

$$c = \ln \left(\frac{1}{6,400,000,000} \right) \rightarrow$$

$$\ln S = 5 \ln(200-t) + \ln \left(\frac{1}{6,400,000,000} \right) \rightarrow$$

$$\ln S = \ln(200-t)^5 + \ln \left(\frac{1}{6,400,000,000} \right) \rightarrow$$

$$e \quad e$$

$$S = e^{\ln(200-t)^5} e^{\ln \left(\frac{1}{6,400,000,000} \right)} \rightarrow$$

$$S = \frac{1}{6,400,000,000} (200 - t)^5$$

a.) Tank loses $5 - 4 = 1$ gal./min so in 20 minutes there are $200 - 20 = 180$ gal. of solution.

b.) $t = 20$ min. $\rightarrow S \approx 29.52$ lbs.,
 $t = 120$ min. $\rightarrow S \approx 0.512$ lbs.

c.) (See a.) The tank is empty in 200 minutes.

Let S be lbs. of salt in tank at time t .

$$3.) \quad \frac{dS}{dt} = (\text{Rate In}) - (\text{Rate Out})$$

$$= \left(\frac{\frac{1}{2} \text{ lb.}}{\text{gal.}} \right) \left(\frac{6 \text{ gal.}}{\text{min.}} \right) - \left(\frac{S \text{ lbs.}}{100 + 2t \text{ gal.}} \right) \left(\frac{4 \text{ gal.}}{\text{min.}} \right) \rightarrow$$

$$\frac{dS}{dt} = 3 - \frac{4}{100 + 2t} \cdot S \quad \rightarrow$$

$$\boxed{\frac{dS}{dt} + \frac{4}{100 + 2t} \cdot S = 3} \quad (\text{First-order linear});$$

$$\left\{ \mu = e^{\int \frac{4}{100 + 2t} dt} = e^{4 \cdot \frac{1}{2} \ln(100 + 2t)} \right.$$

$$= e^{2 \ln(100 + 2t)} = e^{\ln(100 + 2t)^2}$$

$$= (100 + 2t)^2 \quad \left. \right\} \rightarrow$$

$$(100+2t)^2 \cdot \frac{dS}{dt} + 4(100+2t) \cdot S = 3(100+2t)^2 \rightarrow$$

$$D[(100+2t)^2 \cdot S] = 3(100+2t)^2 \rightarrow$$

$$(100+2t)^2 \cdot S = \int 3(100+2t)^2 dt \rightarrow$$

$$(100+2t)^2 \cdot S = \frac{1}{2}(100+2t)^3 + C \rightarrow$$

$$\left\{ \begin{array}{l} t=0, S=10 \text{ lbs.} \rightarrow \\ 100^2(10) = \frac{1}{2}(100)^3 \rightarrow \\ C = -40(100)^2 = -400,000 \end{array} \right\} \rightarrow$$

$$(100+2t)^2 \cdot S = \frac{1}{2}(100+2t)^3 - 400,000 \rightarrow$$

$$S = \frac{1}{2}(100+2t) - \frac{400,000}{(100+2t)^2} ;$$

a.) The tank gains $6 - 4 = 2$ gal./min

i.) $t = 10$ min : $100 + 2(10) = 120$ gal.

ii.) $t = 60$ min : $100 + 2(60) = 220$ gal.

b.) i.) $t = 10$ min :

$$S = \frac{1}{2}(120) - \frac{400,000}{(120)^2} \approx 32.22 \text{ lbs.}$$

ii.) $t = 60$ min :

$$S = \frac{1}{2}(220) - \frac{400,000}{(220)^2} \approx 101.74 \text{ lbs.}$$

c.) $S = 40 \text{ lbs} :$

$$40 = \frac{1}{2}(100 + 2t) - \frac{400,000}{(100 + 2t)^2} \rightarrow$$

$$40 = 50 + t - \frac{400,000}{(100 + 2t)^2} \rightarrow$$

$$\frac{400,000}{(100 + 2t)^2} = 10 + t \rightarrow \text{(using a calculator)}$$

$$t \approx 14.23 \text{ min.}$$

4.) Let S be the gallons of alcohol in the tank at time t . Then

$$\frac{dS}{dt} = (\text{Rate In}) - (\text{Rate Out})$$

$$= \left(\frac{0.06 \text{ gal. alc.}}{1 \text{ gal. beer}} \right) \cdot \left(\frac{3 \text{ gal. beer}}{\text{min.}} \right)$$

$$- \left(\frac{S \text{ gal. alc.}}{400 - t \text{ gal. beer}} \right) \left(\frac{4 \text{ gal. beer}}{\text{min.}} \right)$$

$$\rightarrow \frac{dS}{dt} = 0.18 - \frac{4}{400 - t} \cdot S$$

$$\rightarrow \frac{dS}{dt} + \frac{4}{400 - t} \cdot S = 0.18$$

$$\rightarrow \frac{dS}{dt} + \frac{-4}{t - 400} \cdot S = 0.18 \quad \left(\begin{array}{l} \text{First-Order} \\ \text{Linear} \end{array} \right)$$

$$\left\{ \begin{aligned} \mu &= e^{\int \frac{-4}{t-400} dt} = e^{-4 \ln(t-400)} \\ &= e^{\ln(t-400)^{-4}} = (t-400)^{-4} \end{aligned} \right\} \rightarrow$$

$$(t-400)^{-4} \cdot \frac{dS}{dt} - 4(t-400)^{-5} S = 0.18 (t-400)^{-4} \rightarrow$$

$$D[(t-400)^{-4} \cdot S] = 0.18 (t-400)^{-4} \rightarrow$$

$$(t-400)^{-4} \cdot S = \int 0.18 (t-400)^{-4} dt \rightarrow$$

$$(t-400)^{-4} \cdot S = (0.18) \left(\frac{-1}{3} \right) (t-400)^{-3} + C$$

$$\left\{ \begin{aligned} t=0, \quad S &= (390)(400) = 12 \text{ gal.} \rightarrow \\ \frac{12}{(400)^4} &= \frac{0.06}{(400)^3} + C \rightarrow C = \frac{-12}{400^4} \end{aligned} \right\} \rightarrow$$

$$\frac{12}{(400)^4} = \frac{0.06}{(400)^3} + C \rightarrow C = \frac{-12}{400^4} \rightarrow$$

$$(t-400)^{-4} S = -0.06 (t-400)^{-3} - \frac{12}{400^4} \rightarrow$$

$$S = -0.06 (t-400) - \frac{12}{400^4} (t-400)^4 \rightarrow$$

$$\boxed{S = 24 - 0.06t - \frac{12}{400^4} (t-400)^4} \quad ;$$

a.) The vat loses $4 - 3 = 1$ gal./min.

i.) $t = 10$ min.: $400 - 10(1) = 390$ gal.

ii.) $t = 60$ min.: $400 - 60(1) = 340$ gal.

iii.) $t = 400$ min.: $400 - 400(1) = 0$ gal.

b.) i.) $t = 10$ min.:

$$S = 24 - 0.6 - \frac{12}{400^4} (-390)^4 \approx 12.56 \text{ gal. alc.},$$

390 gal. beer so % of alcohol is

$$\frac{12.56}{390} \approx 3.22 \%$$

ii.) $t = 60$ min.:

$$S = 24 - 3.6 - \frac{12}{400^4} (-340)^4 \approx 14.14 \text{ gal. alc.},$$

340 gal. beer so % of alcohol is

$$\frac{14.14}{340} \approx 4.16 \%$$

$$c.) 4\% = \frac{\text{gal. alc.}}{\text{gal. beer}} = \frac{24 - 0.06t - \frac{12}{400^4} (t-400)^4}{400 - t}$$

$$\rightarrow 16 - 0.04t = 24 - 0.06t - \frac{12}{400^4} (t-400)^4$$

$$\rightarrow \frac{12}{400^4} (t-400)^4 = 8 - 0.02t \rightarrow \text{(using a calculator)}$$

$$t \approx 50.57 \text{ min.}$$