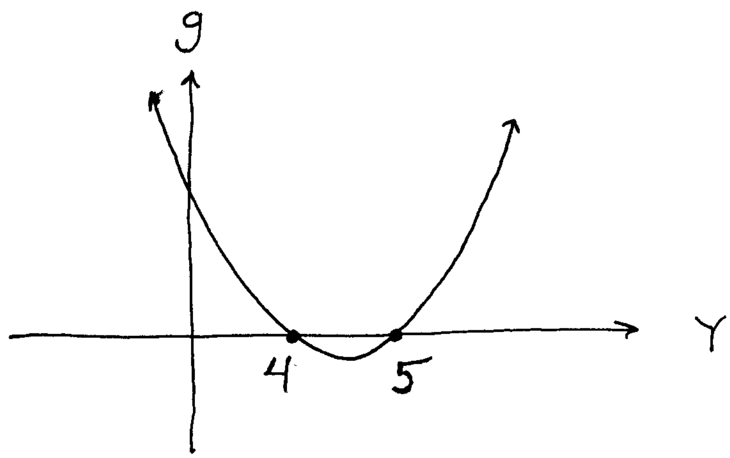
Section 8.2

2.) $\frac{dY}{dX} = (4-Y)(5-Y)$

a.) equilibria : $Y=4$ and $Y=5$

b.) $\frac{dY}{dX} = (4-Y)(5-Y) = g(Y)$



$$\begin{array}{ccccc} + & 0 & - & 0 & + \\ & Y=4 & & Y=5 & \end{array} \qquad g(Y)$$

$Y=4$: Stable
$Y=5$: Unstable

c.) $g'(Y) = (4-Y)(-1) + (-1)(5-Y) \rightarrow$

$g'(Y) = -4 + Y - 5 + Y \rightarrow \underline{g'(Y) = 2Y-9}$;

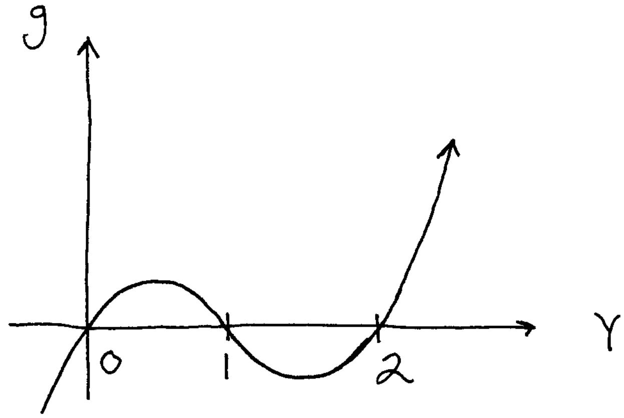
eigenvalue for $Y=4$ is $g'(4) = -1 < 0$, so $Y=4$ is Stable ;

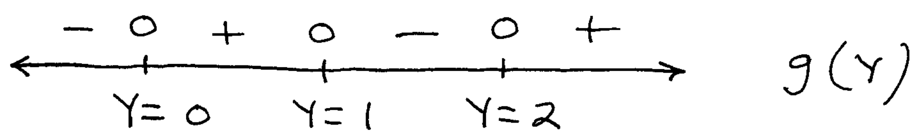
eigenvalue for $Y=5$ is $g'(5) = 1 > 0$, so $Y=5$ is Unstable.

3.) $\frac{dY}{dX} = Y(Y-1)(Y-2)$

a.) equilibria : $Y=0$, $Y=1$, and $Y=2$

b.) $\frac{dY}{dX} = Y(Y-1)(Y-2) = g(Y)$





$Y=0$: Unstable

$Y=1$: Stable

$Y=2$: Unstable

c.) $g'(Y) = (1)(Y-1)(Y-2) + Y(1)(Y-2) + Y(Y-1)(1)$

$= Y^2 - 3Y + 2 + Y^2 - 2Y + Y^2 - Y \longrightarrow$

$g'(Y) = 3Y^2 - 6Y + 2$;

eigenvalue for $Y=0$ is $g'(0) = 2 > 0$, so $Y=0$ is Unstable ;

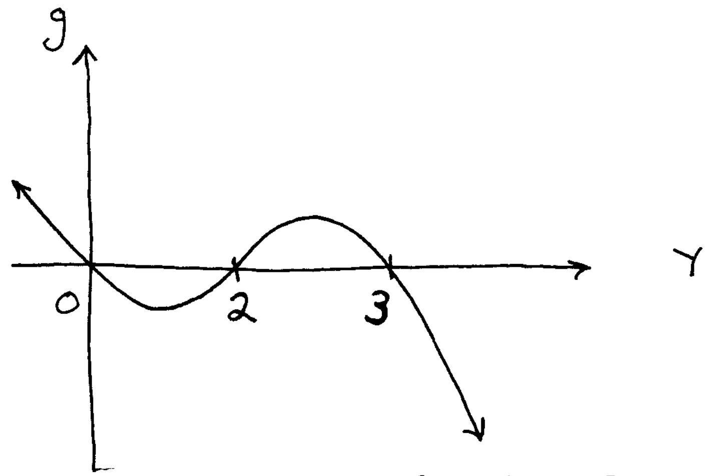
eigenvalue for $Y=1$ is $g'(1) = -1 < 0$, so $Y=1$ is Stable ;

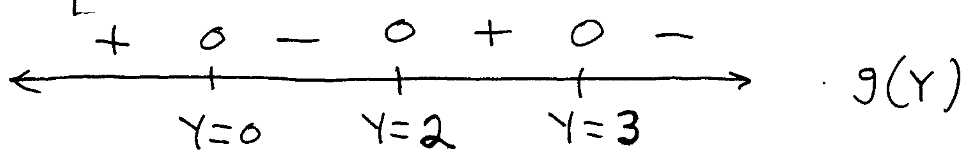
eigenvalue for $Y=2$ is $g'(2) = 2 > 0$, so $Y=2$ is Unstable.

4.) $\dfrac{dY}{dX} = Y(2-Y)(Y-3)$

a.) equilibria : $Y=0$, $Y=2$, and $Y=3$

b.) $\dfrac{dY}{dX} = Y(2-Y)(Y-3) = g(Y)$





$Y=0$: Stable
$Y=2$: Unstable
$Y=3$: Stable

c.) $g'(Y) = (1)(2-Y)(Y-3) + Y(-1)(Y-3) + Y(2-Y)(1)$

$= -Y^2 + 5Y - 6 - Y^2 + 3Y + 2Y - Y^2$

$= -3Y^2 + 10Y - 6$;

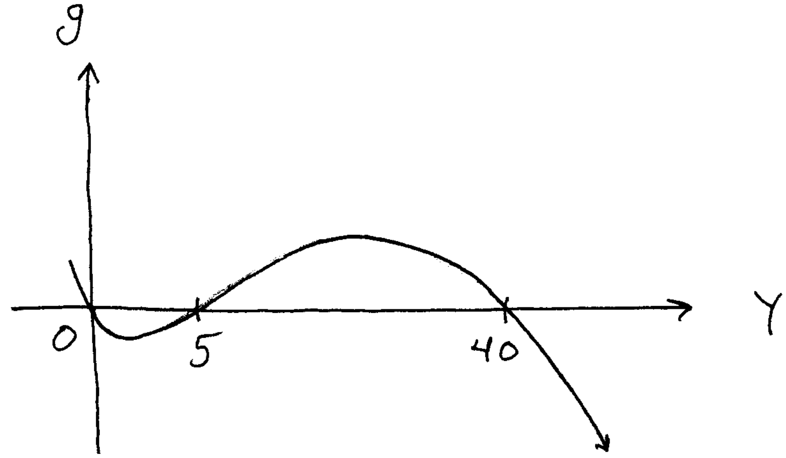
eigenvalue for $Y=0$ is $g'(0) = -6 < 0$, so $Y=0$ is Stable ;

eigenvalue for $Y=2$ is $g'(2) = 2 > 0$, so $Y=2$ is Unstable ;

eigenvalue for $Y=3$ is $g'(3) = -3 < 0$, so $Y=3$ is Stable.

6.) $\frac{dN}{dt} = N\left(1-\frac{N}{50}\right) - \frac{9N}{N+5}$

a.) $g(N) = N\left(1-\frac{N}{50}\right) - \frac{9N}{N+5}$



b.) Set $g(N) = 0$ and solve for N:

$$g(N) = N\left(1-\frac{N}{50}\right) - \frac{9N}{N+5} = N\left[1 - \frac{N}{50} - \frac{9}{N+5}\right]$$

$$= N\left[\frac{50}{50}\cdot\frac{N+5}{N+5} - \frac{N}{50}\cdot\frac{N+5}{N+5} - \frac{50}{50}\cdot\frac{9}{N+5}\right]$$

$$= N \cdot \frac{(50N + 250 - N^2 - 5N - 450)}{50(N+5)}$$

$$= N \cdot \frac{-N^2 + 45N - 200}{50(N+5)} = 0 \rightarrow N = 0 \text{ or}$$

$$-N^2 + 45N - 200 = 0 \rightarrow$$

$$N = \frac{-45 \pm \sqrt{(45)^2 - 4(-1)(-200)}}{-2}$$

$$= \frac{-45 \pm 35}{-2} = 40 \text{ or } 5, \text{ so}$$

equilibria: $N=0$, $N=5$, and $N=40$

c.) $\quad + \quad 0 \quad - \quad 0 \quad + \quad 0 \quad -$ $\quad g(N)$

$\qquad\quad N=0 \qquad N=5 \qquad\quad N=40$

$N=0$: Stable
$N=5$: Unstable
$N=40$: Stable

8.) $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$, $\quad N(0)=N_0$
and carrying capacity (saturation value) is K

a.) $N''(t) = \frac{d}{dt}\left\{rN\left(1-\frac{N}{K}\right)\right\}$

$$= rN\cdot\frac{-1}{K}\cdot N'(t) + r\cdot N'(t)\cdot\left(1-\frac{N}{K}\right)$$

$$= r\cdot N'(t)\cdot\left\{\frac{-N}{K} + 1 - \frac{N}{K}\right\}$$

$$= r\cdot N'(t)\cdot\left\{1-\frac{2N}{K}\right\} = 0 \rightarrow$$

$1-\frac{2N}{K} = 0 \rightarrow N = \frac{1}{2}K$, i.e.
inflection point occurs when

$$N = \frac{1}{2}K.$$

9.) $\frac{dN}{dt} = 2N\left(1-\frac{N}{1000}\right) - H = g(N)$

a.) If $H = 100 \longrightarrow 2N\left(1-\frac{N}{1000}\right) - 100 = 0$

$\rightarrow 2N - \frac{1}{500}N^2 - 100 = 0$

$\rightarrow 1000N - N^2 - 50{,}000 = 0$

$\rightarrow 0 = N^2 - 1000N + 50{,}000$

$$\rightarrow N = \frac{-(-1000) \pm \sqrt{(-1000)^2 - 4(1)(50{,}000)}}{2(1)}$$

$$= \frac{1000 \pm \sqrt{800{,}000}}{2} \rightarrow$$

$N \approx 947.21$ or $N \approx 52.79$

$-$	0	$+$	0	$-$	$g(N)$
	$N \approx 52.79$ (UNSTABLE)		$N \approx 947.21$ (STABLE)		

b.) $2N\left(1-\frac{N}{1000}\right) - H = 0 \longrightarrow$

$2N - \frac{1}{500}N^2 - H = 0 \rightarrow 1000N - N^2 - 500H = 0 \rightarrow$

$0 = N^2 - 1000N + 500H \longrightarrow$

$$N = \frac{-(-1000) \pm \sqrt{(-1000)^2 - 4(1)(500H)}}{2(1)}$$

$$= \frac{1000 \pm \sqrt{2000(500-H)}}{2}$$

$$= \frac{1000 \pm 20\sqrt{5(500-H)}}{2}$$

$$= 500 \pm 10\sqrt{5(500-H)} \longrightarrow$$

$$N = 500 + 10\sqrt{5(500-H)} \quad \text{or}$$

$$N = 500 - 10\sqrt{5(500-H)} \; ;$$

we can see here that H must satisfy $0 \le H \le 500$;

$\boxed{\text{if } H = 500}$, then $N = 500$ and

$$\begin{array}{ccccc} - & & 0 & & - \\ \hline & & N=500 & & \end{array} \quad g(N) = \frac{dN}{dt}$$

so $\frac{dN}{dt}$ is $(-)$ meaning the # of fish N is $(\downarrow)$ (NOT GOOD) ;

$\boxed{\text{if } 0 \le H < 500}$, then:

$$\begin{array}{ccccc} - & 0 & + & 0 & - \\ \hline & \underbrace{N = 500 - 10\sqrt{5(500-H)}}_{\alpha} & & \underbrace{N = 500 + 10\sqrt{5(500-H)}}_{\beta} & \end{array} \quad g(N) = \frac{dN}{dt}$$

so $\frac{dN}{dt}$ is $(+)$ for $\alpha < N < \beta$ meaning the # of fish N is $(\uparrow)$ (GOOD).

10.) $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - hN \quad (r=2,\ K=1000)$

$\rightarrow \frac{dN}{dt} = 2N\left(1-\frac{N}{1000}\right) - hN = g(N)$

a.) $g(N) = 0 \rightarrow \quad (h = 0.1)$

$2N\left(1-\frac{N}{1000}\right) - 0.1N = 0 \rightarrow$

$N\left\{2\left(1-\frac{N}{1000}\right) - 0.1\right\} = 0 \rightarrow$

$N\left\{2 - \frac{1}{500}N - \frac{1}{10}\right\} = 0 \rightarrow$

$\boxed{N=0}$ or $2 - \frac{1}{500}N - \frac{1}{10} = 0 \rightarrow$

$\frac{19}{10} = \frac{1}{500}N \rightarrow \boxed{N = 950}$

$-\quad 0 \quad + \quad 0 \quad -\qquad g(N)$

$N=0 \qquad\qquad N=950$

$\uparrow \qquad\qquad \uparrow$

$\boxed{\text{UNSTABLE}} \qquad \boxed{\text{STABLE}}$;

to find the maximum harvest rate, $N'(t)$, we need a sign chart for $\frac{d}{dt}(N'(t)) = N''(t)$:

$N'(t) = 2N - \frac{1}{500}N^2 - \frac{1}{10}N$

$= \frac{19}{10}N - \frac{1}{500}N^2 \xrightarrow{D}$

$$N''(t) = \frac{19}{10} \cdot N'(t) - \frac{1}{500} \cdot 2N \cdot N'(t)$$

$$= N'(t) \cdot \left\{ \frac{19}{10} - \frac{1}{250} N \right\} = 0 \rightarrow$$

$$\frac{19}{10} - \frac{1}{250} N = 0 \longrightarrow N = 475$$

$$\begin{array}{ccc} + & 0 & - \end{array} \qquad N''(t)$$

$N = 475$ fish

so $\underline{\text{MAXIMUM}}$ rate, $N'(t)$, occurs when

$\boxed{N = 475}$ and rate

$$N'(t) = 2(475) - \frac{1}{500}(475)^2 - \frac{1}{10}(475) \rightarrow$$

$\boxed{N'(t) \approx 451.25 \text{ fish/time units}}$

b.) assume $h < r = 2$, then

$$g(N) = 2N - \frac{1}{500} N^2 - hN = 0 \rightarrow$$

$$N \left\{ 2 - \frac{1}{500} N - h \right\} = 0 \rightarrow$$

$$2 - \frac{1}{500} N - h = 0 \rightarrow 2 - h = \frac{1}{500} N \rightarrow$$

$\boxed{N = 1000 - 500h}$, nontrivial equilibrium c.) i.) (eigenvalue approach)

$g'(N) = 2 - \frac{1}{250}N - h$, and

$g'(1000-500h) = 2 - \frac{1}{250}(1000-500h) - h$

$= 2 - 4 + 2h - h = h - 2 < 0$, so

$N = 1000 - 500h$ is <u>STABLE</u>.

ii.) (graphical approach)

$g(N) = 2N - \frac{1}{500}N^2 - hN$

$= N\left\{(2-h) - \frac{1}{500}N\right\}$

$+ \quad 0 \quad -$ $\qquad g(N)$

$N = 1000 - 500h$

↑

<u>STABLE</u>

14.) a.) 1000 l. H_2O, 2 kg = 2000 g. salt,
so concentration of salt is

$$\frac{2000 \text{ g.}}{1000 \text{ l.}} = 2 \text{ g./l.}$$

b.) X : # l. fresh H_2O pumped in, so
1000 − X : # l. salt H_2O in tank; then

total kg. of salt in tank is

$$\underset{l.}{(1000 - x)} \cdot \underset{g/l.}{(2)} = \underset{l.}{(1000)}\underset{g./l.}{(1)} \rightarrow$$

$$2000 - 2x = 1000 \rightarrow 2x = 1000 \rightarrow$$

$x = 500$ l. of fresh H_2O

c.) Let t : # seconds ; then

pump 1: $x = t$ l. fresh H_2O

pump 2: $x = 2t$ l. fresh H_2O ; so

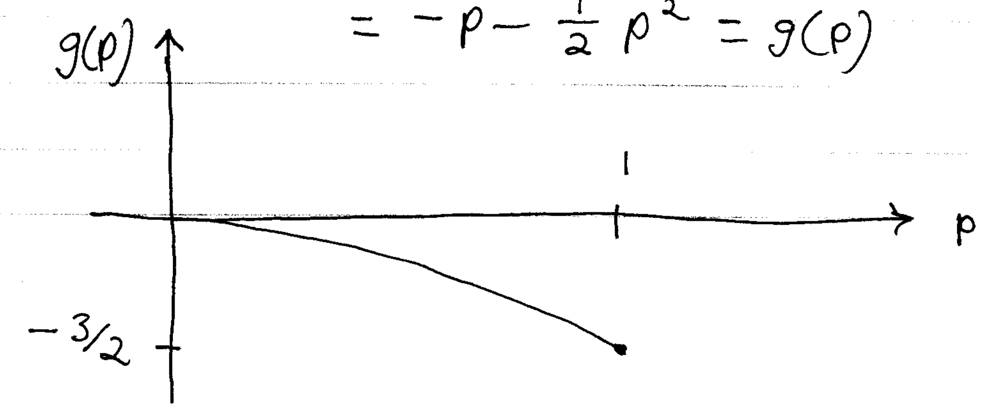
pump 1: $x = t = 500$ sec.

pump 2: $x = 2t = 500 \rightarrow t = 250$ sec.

21.) $\frac{dp}{dt} = 0.5\,p(1-p) - 1.5\,p$

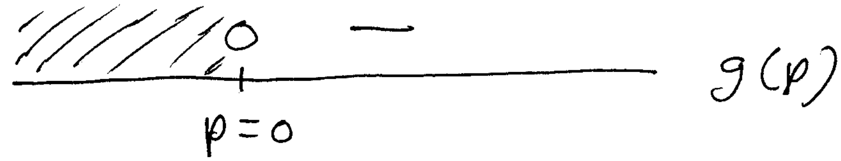
$= -p - \frac{1}{2}p^2 = g(p)$

a.)



b.) $g(p)=0 \rightarrow -p-\frac{1}{2}p^2 = -p\left(1+\frac{1}{2}p\right)=0$

$\rightarrow$ $p=0$ on interval $[0,1]$:



$\uparrow$

$\boxed{\text{STABLE}}$

c.) $g(p) = -p-\frac{1}{2}p^2 \xrightarrow{D}$

$g'(p) = -1-p$ and

$g'(0) = -1-(0) = -1 < 0$, so

$\boxed{p=0}$ is STABLE .

24.) $\frac{dN}{dt} = 2N(N-10)\left(1-\frac{N}{100}\right) = g(N)$

a.) $g(N)=0 \rightarrow$ equilibria are

$\boxed{N=0}$, $\boxed{N=10}$, $\boxed{N=100}$

b.) $g'(N) = 2(N-10)\left(1-\frac{N}{100}\right)$

$+ 2N\cdot(1)\cdot\left(1-\frac{N}{100}\right) + 2N(N-10)\cdot\left(\frac{-1}{100}\right)$;

$g'(0) = 2(-10)(1) + 0 + 0 = -20 < 0$, so

$\boxed{N=0}$ is STABLE ;

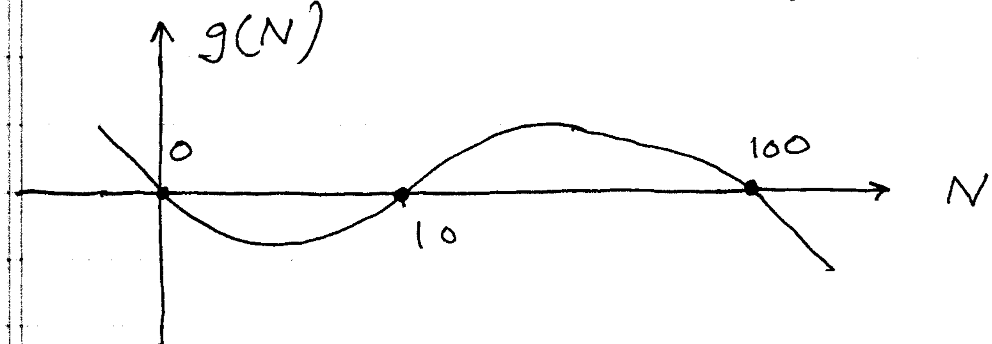
$g'(10) = 0 + 20\cdot(0.9) + 0 = 18 > 0$, so

$\boxed{N=10}$ is UNSTABLE ;

$g'(100) = 0 + 0 + 2(100)(90)\left(\frac{-1}{100}\right) = -180 < 0$,

so $\boxed{N=100}$ is STABLE .

c.) $g(N) = 2N(N-10)\left(1-\frac{N}{100}\right)$



+	0	−	0	+	0	−	$g(N)$
	$N=0$		$N=10$		$N=100$		
	↑		↑		↑		
	STABLE		UNSTABLE		STABLE		

25.) $\frac{dN}{dt} = 0.3N(N-17)\left(1-\frac{N}{200}\right) = g(N)$

a.) $g(N) = 0 \rightarrow$ equilibria are

$\boxed{N=0}$, $\boxed{N=17}$, $\boxed{N=200}$

b.) $g'(N) = (0.3)(N-17)\left(1-\frac{N}{200}\right)$
$+ (0.3N)(1)\left(1-\frac{N}{200}\right) + (0.3N)(N-17)\left(\frac{-1}{200}\right)$;

$g'(0) = (0.3)(-17) + 0 + 0 = -5.1 < 0$,
so $\boxed{N=0}$ is STABLE;
$g'(17) = 0 + (5.1)\left(\frac{183}{200}\right) + 0 = 4.6665 > 0$,
so $\boxed{N=17}$ is UNSTABLE;
$g'(200) = 0 + 0 + (60)(183)\left(\frac{-1}{200}\right) = -54.9 < 0$,
so $\boxed{N=200}$ is STABLE.

c.) $g(N) = 0.3N(N-17)\left(1 - \frac{N}{200}\right)$