Worksheet 5

1.) $Y' = 2t + 3$ and $Y(0) = 4 \rightarrow$

$\mathcal{L}\{Y'\} = \mathcal{L}\{2t+3\} \rightarrow$

$s\mathcal{L}\{Y\} - \underbrace{Y(0)}_{4} = 2 \cdot \frac{1}{s^2} + 3 \cdot \frac{1}{s} \rightarrow$

$\mathcal{L}\{Y\} = 2 \cdot \frac{1}{s^3} + 3 \cdot \frac{1}{s^2} + 4 \cdot \frac{1}{s} \rightarrow$

$\mathcal{L}\{Y\} = \frac{2}{s^3} + 3 \cdot \frac{1}{s^2} + 4 \cdot \frac{1}{s} \rightarrow$

$Y = t^2 + 3t + 4$

2.) $Y' = e^t$ and $Y(0) = 2 \rightarrow$

$\mathcal{L}\{Y'\} = \mathcal{L}\{e^t\} \rightarrow$

$s\mathcal{L}\{Y\} - \underbrace{Y(0)}_{2} = \frac{1}{s-1} \rightarrow$

$\mathcal{L}\{Y\} = \frac{2}{s} + \frac{1}{s(s-1)}$

$\left(\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} \rightarrow A(s-1) + Bs = 1 :\right.$

$\underline{\text{Let } s = 0}: -A = 1 \rightarrow A = -1,$

$\left.\underline{\text{Let } s = 1}: B = 1 \right) \rightarrow$

$\mathcal{L}\{Y\} = \frac{2}{s} + \left(\frac{-1}{s} + \frac{1}{s-1} \right) \rightarrow$

$\mathcal{L}\{Y\} = \frac{1}{s} + \frac{1}{s-1} \rightarrow$ $Y = 1 + e^t$

3.) $Y' = t + \sin t$ and $Y(0) = 0 \rightarrow$

$\mathcal{L}\{Y'\} = \mathcal{L}\{t + \sin t\} \rightarrow$

$s\mathcal{L}\{Y\} - \underbrace{Y(0)}_{0} = \frac{1}{s^2} + \frac{1}{s^2+1} \rightarrow$

$$\mathcal{L}\{Y\} = \frac{1}{s^3} + \frac{1}{s(s^2+1)}$$

$$\left(\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \rightarrow\right.$$

$A(s^2+1) + (Bs+C)s = 1$:

Let $s=0$: $A=1$

Let $s=i$: $A(0) + (Bi+C)i = 1 \rightarrow$

$(-B)+(C)i = (1)+(0)i \rightarrow -B=1 \rightarrow$

$\left. B=-1 \text{ and } C=0 \right) \rightarrow$

$$\mathcal{L}\{Y\} = \frac{1}{s^3} + \frac{1}{s} - \frac{s}{s^2+1} \rightarrow$$

$$\mathcal{L}\{Y\} = \frac{1}{2}\cdot\frac{2}{s^3} + \frac{1}{s} - \frac{s}{s^2+1} \rightarrow$$

$$Y = \frac{1}{2}t^2 + 1 - \cos t$$

4.) $Y' = te^t$ and $Y(0)=1 \rightarrow$

$\mathcal{L}\{Y'\} = \mathcal{L}\{te^t\} \rightarrow$

$$s\mathcal{L}\{Y\} - \underbrace{Y(0)}_{1} = \frac{1}{(s-1)^2} \rightarrow$$

$$\mathcal{L}\{Y\} = \frac{1}{s} + \frac{1}{s(s-1)^2}$$

$$\left(\frac{1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \rightarrow\right.$$

$A(s-1)^2 + Bs(s-1) + Cs = 1$:

Let $s=1$: $C=1$,

Let $s=0$: $A=1$,

Let $s=2$: $(1)(1)+B(2)(1)+(1)(2)=1 \rightarrow$

$2B=-2 \rightarrow B=-1$) $\rightarrow$

$$\mathcal{L}\{Y\}=\frac{1}{s}+\left(\frac{1}{s}+\frac{-1}{s-1}+\frac{1}{(s-1)^2}\right) \rightarrow$$

$$\mathcal{L}\{Y\}=\frac{2}{s}-\frac{1}{s-1}+\frac{1}{(s-1)^2} \rightarrow$$

$$Y=2-e^t+te^t$$

5.) $Y''=6$ and $Y(0)=1,\ Y'(0)=2 \rightarrow$

$$\mathcal{L}\{Y''\}=\mathcal{L}\{6\} \rightarrow$$

$$s^2\mathcal{L}\{Y\}-s\underset{1}{Y(0)}-\underset{2}{Y'(0)}=6/s \rightarrow$$

$$s^2\mathcal{L}\{Y\}=2+s+\frac{6}{s} \rightarrow$$

$$\mathcal{L}\{Y\}=\frac{2}{s^2}+\frac{1}{s}+\frac{6}{s^3} \rightarrow$$

$$\mathcal{L}\{Y\}=2\cdot\frac{1}{s^2}+\frac{1}{s}+3\cdot\frac{2}{s^3} \rightarrow$$

$$Y=2t+1+3t^2$$

6.) $Y''=6t$ and $Y(0)=0,\ Y'(0)=-1 \rightarrow$

$$\mathcal{L}\{Y''\}=\mathcal{L}\{6t\} \rightarrow$$

$$s^2\mathcal{L}\{Y\}-s\cdot\underset{0}{Y(0)}-\underset{-1}{Y'(0)}=6\cdot\frac{1}{s^2} \rightarrow$$

$$s^2\mathcal{L}\{Y\}=\frac{6}{s^2}-1 \rightarrow$$

$$\mathcal{L}\{Y\}=\frac{6}{s^4}-\frac{1}{s^2} \rightarrow$$

$$\mathcal{L}\{Y\} = \frac{3!}{s^4} - \frac{1}{s^2} \rightarrow \boxed{Y = t^3 - t}$$

7.) $Y'' = \cos t$ and $Y(0) = 0,\ Y'(0) = 0 \rightarrow$

$$\mathcal{L}\{Y''\} = \mathcal{L}\{\cos t\} \rightarrow$$

$$s^2\mathcal{L}\{Y\} - s\underbrace{Y(0)}_{0} - \underbrace{Y'(0)}_{0} = \frac{s}{s^2+1} \rightarrow$$

$$\mathcal{L}\{Y\} = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

(See solution to problem 3.) $\rightarrow$

$$\mathcal{L}\{Y\} = \frac{1}{s} - \frac{s}{s^2+1} \longrightarrow$$

$$\boxed{Y = 1 - \cos t}$$

8.) $Y'' - Y' - 2Y = 0$ and $Y(0) = 5,\ Y'(0) = 1 \rightarrow$

$$\mathcal{L}\{Y'' - Y' - 2Y\} = \mathcal{L}\{0\} \longrightarrow$$

$$\mathcal{L}\{Y''\} - \mathcal{L}\{Y'\} - \mathcal{L}\{2Y\} = 0 \longrightarrow$$

$$(s^2\mathcal{L}\{Y\} - sY(0) - Y'(0)) - (s\mathcal{L}\{Y\} - Y(0))$$
$$- 2\mathcal{L}\{Y\} = 0 \rightarrow$$

$$s^2\mathcal{L}\{Y\} - 5s - 1 - s\mathcal{L}\{Y\} + 5 - 2\mathcal{L}\{Y\} = 0 \rightarrow$$

$$(s^2 - s - 2)\mathcal{L}\{Y\} = 5s - 4 \longrightarrow$$

$$\mathcal{L}\{Y\} = \frac{5s-4}{s^2-s-2} = \frac{5s-4}{(s-2)(s+1)}$$

$$= \frac{A}{s-2} + \frac{B}{s+1} \rightarrow A(s+1) + B(s-2) = 5s - 4 :$$

Let $s=2$: $3A=6 \to A=2$,
Let $s=-1$: $-3B=-9 \to B=3$; then

$$\mathcal{L}\{Y\} = \frac{2}{s-2} + \frac{3}{s+1} \longrightarrow$$

$$Y = 2e^{2t} + 3e^{-t}$$

9.) $Y'' + Y' = 1$ and $Y(0)=2$, $Y'(0)=3 \longrightarrow$

$$\mathcal{L}\{Y''+Y'\} = \mathcal{L}\{1\} \longrightarrow$$

$$\mathcal{L}\{Y''\} + \mathcal{L}\{Y'\} = 1/s \longrightarrow$$

$$s^2\mathcal{L}\{Y\} - sY(0) - Y'(0) + s\mathcal{L}\{Y\} - Y(0) = 1/s \longrightarrow$$

$$s^2\mathcal{L}\{Y\} - 2s - 3 + s\mathcal{L}\{Y\} - 2 = 1/s \longrightarrow$$

$$(s^2+s)\mathcal{L}\{Y\} = 2s + 5 + 1/s \longrightarrow$$

$$\mathcal{L}\{Y\} = \frac{2s}{s^2+s} + \frac{5}{s^2+s} + \frac{1}{s(s^2+s)} \longrightarrow$$

$$\mathcal{L}\{Y\} = \frac{2\cancel{s}}{\cancel{s}(s+1)} + \frac{5s}{s(s^2+s)} + \frac{1}{s(s^2+s)} \longrightarrow$$

$$\mathcal{L}\{Y\} = \frac{2}{s+1} + \frac{5s+1}{s^2(s+1)}$$

$$\left(\frac{5s+1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} \longrightarrow\right.$$

$As(s+1) + B(s+1) + Cs^2 = 5s+1$:

Let $s=0$: $B=1$,
Let $s=-1$: $C=-4$,
Let $s=1$: $2A + 2(1) + (-4)(1) = 6 \to$

$$2A = 8 \rightarrow A = 4 \Big) \longrightarrow$$

$$\mathcal{L}\{Y\} = \frac{2}{s+1} + \left(\frac{4}{s} + \frac{1}{s^2} + \frac{-4}{s+1}\right) \longrightarrow$$

$$\mathcal{L}\{Y\} = \frac{4}{s} + \frac{1}{s^2} - \frac{2}{s+1} \longrightarrow$$

$$Y = 4 + t - 2e^{-t}$$