

Section 10.5

2.) $f(x, y) = e^x \sin y$, $x = t$, $y = t^3 \xrightarrow{D}$

$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= e^x \sin y \cdot (1) + e^x \cos y \cdot (3t^2)$$

and $t = 1 \rightarrow x = 1, y = 1$ so

$$\frac{dw}{dt} = e^1 \sin 1 + e^1 \cos 1 \cdot (3) = e \sin 1 + 3e \cos 1$$

3.) $f(x, y) = \sqrt{x^2 + y^2}$, $x = t$, $y = \sin t \xrightarrow{D}$

$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot 2x \cdot (1) + \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot 2y \cdot \cos t$$

and $t = \frac{\pi}{3} \rightarrow x = \frac{\pi}{3}, y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ so

$$\frac{dw}{dt} = \frac{\frac{\pi}{3}}{\sqrt{\frac{\pi^2}{9} + \frac{3}{4}}} + \frac{\frac{\sqrt{3}}{2} \cos^{\frac{1}{2}} \frac{\pi}{3}}{\sqrt{\frac{\pi^2}{9} + \frac{3}{4}}} = \frac{\frac{\pi}{3} + \frac{\sqrt{3}}{4}}{\sqrt{\frac{\pi^2}{9} + \frac{3}{4}}}$$

4.) $f(x, y) = \ln(xy - x^2)$, $x = t^2$, $y = t \xrightarrow{D}$

$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= \frac{y - 2x}{xy - x^2} \cdot (2t) + \frac{x}{xy - x^2} \cdot (1) \quad \text{and}$$

$t = 5 \rightarrow x = 25, y = 5$ so

$$\frac{dw}{dt} = \frac{5 - 50}{125 - 625} (10) + \frac{25}{125 - 625}$$

$$= \frac{-450}{-500} + \frac{25}{-500} = \frac{425}{500} = 0.85$$

$$7.) z = f(x, y), \quad x = u(t), \quad y = v(t) \xrightarrow{D}$$

$$\frac{dz}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} = f_x \cdot u'(t) + f_y \cdot v'(t)$$

$$8.) w = e^{f(x, y)}, \quad x = u(t), \quad y = v(t) \xrightarrow{D}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= e^{f(x, y)} \cdot f_x \cdot u'(t) + e^{f(x, y)} \cdot f_y \cdot v'(t)$$

$$11.) \ln(x^2 + y^2) = 3xy \rightarrow$$

$$F(x, y) = \ln(x^2 + y^2) - 3xy = 0 \xrightarrow{D}$$

$$F_x = \frac{2x}{x^2 + y^2} - 3y, \quad F_y = \frac{2y}{x^2 + y^2} - 3x \quad \text{so}$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{3y - \frac{2x}{x^2 + y^2}}{\frac{2y}{x^2 + y^2} - 3x} \cdot \frac{x^2 + y^2}{x^2 + y^2}$$

$$= \frac{3y(x^2 + y^2) - 2x}{2y - 3x(x^2 + y^2)} = \frac{3x^2y + 3y^3 - 2x}{2y - 3x^3 - 3xy^2}$$

$$14.) y = \arctan x \rightarrow F(x, y) = y - \arctan x = 0$$

$$\xrightarrow{D} F_x = \frac{-1}{1+x^2}, \quad F_y = 1 \quad \text{so}$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{\frac{1}{1+x^2}}{1} = \frac{1}{1+x^2}$$

$$17.) f(x, y) = x^3 y^2 \xrightarrow{D} f_x = 3x^2 y^2,$$

$$f_y = 2x^3 y \quad \text{so} \quad \nabla f = \begin{bmatrix} 3x^2 y^2 \\ 2x^3 y \end{bmatrix}$$

$$18.) f(x, y) = \frac{xy}{x^2 + y^2} \xrightarrow{D}$$

$$f_x = \frac{(x^2 + y^2) \cdot y - xy \cdot 2x}{(x^2 + y^2)^2} = \frac{x^2 y + y^3 - 2x^2 y}{(x^2 + y^2)^2}$$

$$= \frac{y^3 - x^2 y}{(x^2 + y^2)^2} \quad ;$$

$$f_y = \frac{(x^2 + y^2) \cdot x - xy \cdot (2y)}{(x^2 + y^2)^2} = \frac{x^3 + xy^2 - 2xy^2}{(x^2 + y^2)^2}$$

$$= \frac{x^3 - xy^2}{(x^2 + y^2)^2} \quad \Rightarrow$$

$$\nabla f = \begin{bmatrix} \frac{y^3 - x^2 y}{(x^2 + y^2)^2} \\ \frac{x^3 - xy^2}{(x^2 + y^2)^2} \end{bmatrix}$$

$$22.) f(x, y) = \tan\left(\frac{x-y}{x+y}\right) \xrightarrow{D}$$

$$f_x = \sec^2\left(\frac{x-y}{x+y}\right) \cdot \frac{(x+y)(1) - (x-y)(1)}{(x+y)^2}$$

$$= \sec^2\left(\frac{x-y}{x+y}\right) \cdot \frac{2y}{(x+y)^2} \quad ;$$

$$f_y = \sec^2\left(\frac{x-y}{x+y}\right) \cdot \frac{(x+y)(-1) - (x-y)(1)}{(x+y)^2}$$

$$= \sec^2\left(\frac{x-y}{x+y}\right) \cdot \frac{-2x}{(x+y)^2} \quad \Rightarrow$$

$$\nabla f = \begin{bmatrix} \sec^2\left(\frac{x-y}{x+y}\right) \cdot \frac{2y}{(x+y)^2} \\ \sec^2\left(\frac{x-y}{x+y}\right) \cdot \frac{-2x}{(x+y)^2} \end{bmatrix}$$

25.) $f(x, y) = \sqrt{2x^2 + y^2}$, pt. $(1, 2)$,
 $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $u = \frac{1}{|X|} X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$;

$$f_x = \frac{1}{2} (2x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{2x^2 + y^2}}$$

$$f_y = \frac{1}{2} (2x^2 + y^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{2x^2 + y^2}} ; \text{ then}$$

$$\begin{aligned} D_u f(1, 2) &= \nabla f(1, 2) \cdot u = \begin{bmatrix} f_x(1, 2) \\ f_y(1, 2) \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{2}{2\sqrt{3}} + \frac{2}{2\sqrt{3}} \\ &= \frac{2}{\sqrt{3}} \end{aligned}$$

27.) $f(x, y) = e^{x+y}$, pt. $(0, 0)$, $X = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

so $u = \frac{1}{|X|} X = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$;

$$f_x = e^{x+y}, \quad f_y = e^{x+y} \quad \text{so} \quad \nabla f = \begin{bmatrix} e^{x+y} \\ e^{x+y} \end{bmatrix};$$

then

$$\begin{aligned} D_u f(0, 0) &= \nabla f(0, 0) \cdot u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \\ &= \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2} \end{aligned}$$

28.) $f(x, y) = x^3 y^2$, pt. $(2, 3)$, $X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
 so $u = \frac{1}{|X|} X = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$;

$f_x = 3x^2 y^2$, $f_y = 2x^3 y$ so $\nabla f = \begin{bmatrix} 3x^2 y^2 \\ 2x^3 y \end{bmatrix}$;

then

$$D_u f(2, 3) = \nabla f(2, 3) \cdot u = \begin{bmatrix} 108 \\ 48 \end{bmatrix} \cdot \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$= \frac{-216}{\sqrt{5}} + \frac{48}{\sqrt{5}} = \frac{-168}{\sqrt{5}}$$

31.) $f(x, y) = 2x^2 y - 3x$, pt. $(2, 1)$, vector
 $\overrightarrow{PQ} = \begin{bmatrix} 3-2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$;

$f_x = 4xy - 3$, $f_y = 2x^2$ so

$\nabla f = \begin{bmatrix} 4xy - 3 \\ 2x^2 \end{bmatrix}$; then

$$D_u f(2, 1) = \nabla f(2, 1) \cdot u = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{5}{\sqrt{2}} + \frac{8}{\sqrt{2}} = \frac{13}{\sqrt{2}}$$

34.) $f(x, y) = e^{x-y}$, pt. $(2, 2)$, vector

$\overrightarrow{PQ} = \begin{bmatrix} 1-2 \\ -1-2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ so $u = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$;

$f_x = e^{x-y}$, $f_y = e^{x-y} \cdot (-1) = -e^{x-y}$, so

$$\nabla f = \begin{bmatrix} e^{x-y} \\ -e^{x-y} \end{bmatrix}, \text{ then}$$

$$\begin{aligned} D_u f(2,2) &= \nabla f(2,2) \cdot u = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \\ &= \frac{-1}{\sqrt{10}} + \frac{3}{\sqrt{10}} = \frac{2}{\sqrt{10}}. \end{aligned}$$

$$35.) f(x,y) = 3xy - x^2 \xrightarrow{D} f_x = 3y - 2x, \\ f_y = 3x \quad \text{so} \quad \nabla f = \begin{bmatrix} 3y - 2x \\ 3x \end{bmatrix}; \text{ so at}$$

pt. $(-1, 1)$ f \uparrow most rapidly in the direction of the gradient vector

$$\nabla f(-1, 1) = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

$$37.) f(x,y) = \sqrt{x^2 - y^2} \xrightarrow{D}$$

$$f_x = \frac{1}{2}(x^2 - y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 - y^2}},$$

$$f_y = \frac{1}{2}(x^2 - y^2)^{-1/2} \cdot (-2y) = \frac{-y}{\sqrt{x^2 - y^2}} \quad \text{so}$$

$$\nabla f = \begin{bmatrix} \frac{x}{\sqrt{x^2 - y^2}} \\ \frac{-y}{\sqrt{x^2 - y^2}} \end{bmatrix}; \text{ at pt. } (5, 3) \text{ } f \uparrow$$

most rapidly in

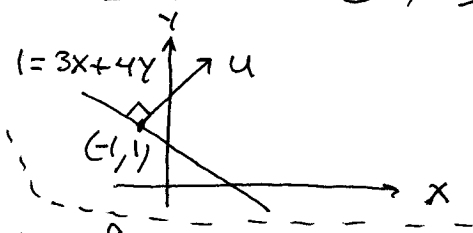
the direction of the gradient vector

$$\nabla f(5, 3) = \begin{bmatrix} \frac{5}{4} \\ -\frac{3}{4} \end{bmatrix}$$

39.) $f(x, y) = 3x + 4y \xrightarrow{D} f_x = 3, f_y = 4$ and
 $\nabla f = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $\nabla f(-1, 1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$;
 $f(-1, 1) = 3(-1) + 4(1) = 1$ so gradient

vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is \perp to level curve
 at pt. $(-1, 1)$

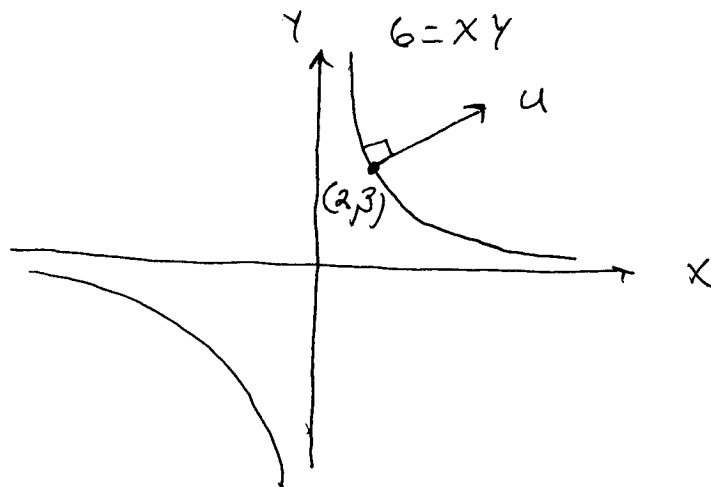
$1 = 3x + 4y$, so unit \perp vector is
 $u = \frac{1}{\sqrt{25}} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$



42.) $f(x, y) = xy \xrightarrow{D} f_x = y, f_y = x$ and
 $\nabla f = \begin{bmatrix} y \\ x \end{bmatrix}$ so $\nabla f(2, 3) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$;

$f(2, 3) = 6$ so gradient vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is
 \perp to level curve $6 = xy$ at pt.
 $(2, 3)$, so unit \perp vector is

$u = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix}$.



$$44.) f(x, y) = x^2 - y^2 \xrightarrow{D} f_x = 2x, f_y = -2y$$

$$\text{so } \nabla f = \begin{bmatrix} 2x \\ -2y \end{bmatrix}; \text{ so at pt. } (2, 3) \text{ } f \downarrow$$

most rapidly in the OPPOSITE
direction of the gradient vector

$$\nabla f(2, 3) = \begin{bmatrix} 4 \\ -6 \end{bmatrix}, \text{ i.e., in the}$$

$$\text{direction of } -\nabla f(2, 3) = \begin{bmatrix} -4 \\ 6 \end{bmatrix}.$$