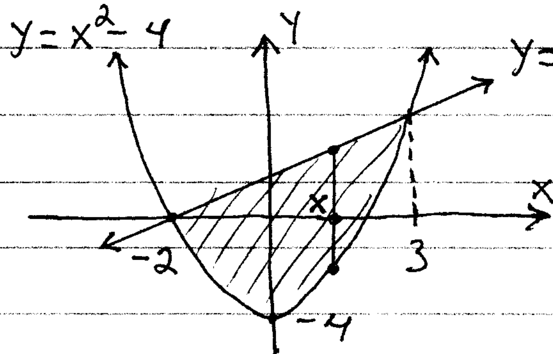
Section 6.3

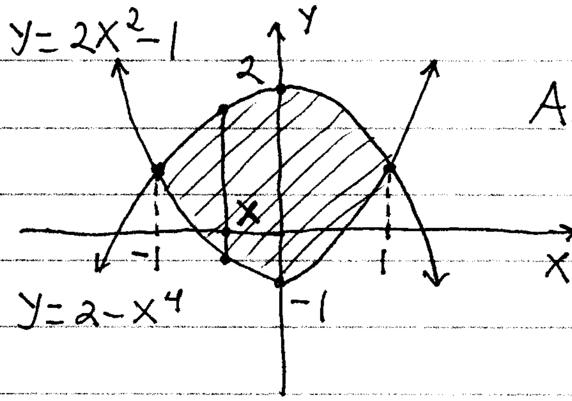
1.) $y=x^2-4,\ y=x+2 \rightarrow x^2-4=x+2 \rightarrow$
$x^2-x-6=0 \longrightarrow (x-3)(x+2)=0 \rightarrow x=3,\ x=-2$



$$\text{Area} = \int_{-2}^{3} (\text{TOP}-\text{BOTTOM})\,dx$$
$$= \int_{-2}^{3} \left[(x+2)-(x^2-4)\right] dx$$

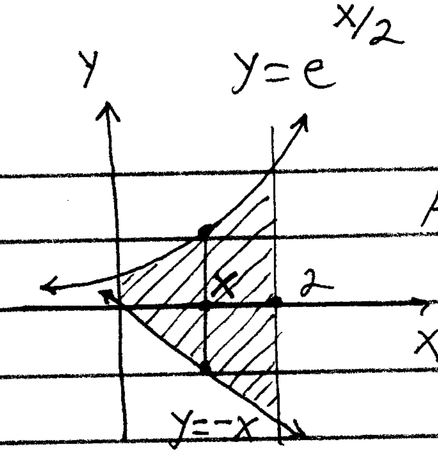
$$= \int_{-2}^{3} \left[6+x-x^2\right] dx = \left(6x+\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\right)\Big|_{-2}^{3}$$
$$= \left(18+\tfrac{9}{2}-9\right)-\left(-12+2+\tfrac{8}{3}\right)$$
$$= 9+\tfrac{9}{2}+10-\tfrac{8}{3} = \tfrac{114}{6}+\tfrac{27}{6}-\tfrac{16}{6}=\tfrac{125}{6}$$

2.) $y=2x^2-1,\ y=2-x^4 \rightarrow 2x^2-1=2-x^4 \longrightarrow$
$x^4+2x^2-3=0 \longrightarrow (x^2-1)(x^2+3)=0 \longrightarrow$
$(x-1)(x+1)(x^2+3)=0 \longrightarrow x=1,\ x=-1$



$$\text{Area} = \int_{-1}^{1} (\text{TOP}-\text{BOTTOM})\,dx$$
$$= \int_{-1}^{1} \left[(2-x^4)-(2x^2-1)\right] dx$$
$$= \int_{-1}^{1} \left[3-x^4-2x^2\right] dx$$

$$= \left(3x-\tfrac{1}{5}x^5-\tfrac{2}{3}x^3\right)\Big|_{-1}^{1} = \left(3-\tfrac{1}{5}-\tfrac{2}{3}\right)-\left(-3+\tfrac{1}{5}+\tfrac{2}{3}\right)$$
$$= 6-\tfrac{2}{5}-\tfrac{4}{3} = \tfrac{90}{15}-\tfrac{6}{15}-\tfrac{20}{15}=\tfrac{64}{15}$$

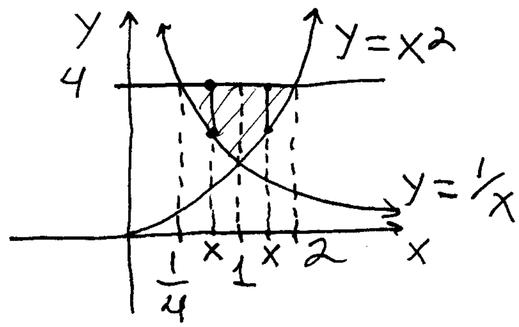
3.) 

$$\text{Area} = \int_0^2 [\text{TOP} - \text{BOTTOM}]\,dx$$

$$= \int_0^2 [e^{x/2} - (-x)]\,dx$$

$$= \int_0^2 (e^{x/2} + x)\,dx$$

$$= \left(2e^{x/2} + \tfrac{1}{2}x^2\right)\Big|_0^2 = (2e+2) - (2e^0+0) = 2e$$

7.) 

$\frac{1}{x} = x^2 \rightarrow x^3 = 1 \rightarrow$
$x = 1$;

$4 = \frac{1}{x} \rightarrow x = \frac{1}{4}$;

$x^2 = 4 \rightarrow x = 2$; then

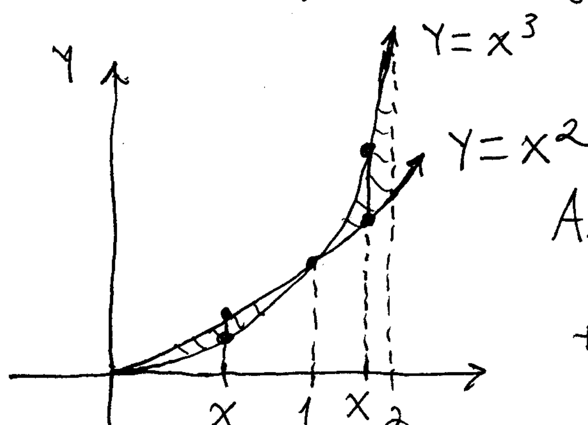
$$\text{Area} = \int_{\frac{1}{4}}^{1} (\text{TOP} - \text{BOTTOM})\,dx + \int_1^2 (\text{TOP} - \text{BOTTOM})\,dx$$

$$= \int_{\frac{1}{4}}^{1} \left(4 - \frac{1}{x}\right) dx + \int_1^2 (4 - x^2)\,dx$$

$$= (4x - \ln x)\Big|_{\frac{1}{4}}^{1} + \left(4x - \tfrac{1}{3}x^3\right)\Big|_1^2$$

$$= (4 - \overset{0}{\cancel{\ln 1}}) - (1 - \ln(1/4)) + \left(8 - \tfrac{8}{3}\right) - \left(4 - \tfrac{1}{3}\right)$$

$$= 3 + \ln(1/4) + 4 - \frac{7}{3} = 7 + \ln(1/4) - \frac{7}{3}$$

11.) 

$x^3 = x^2 \rightarrow x^3 - x^2 = 0 \rightarrow$
$x^2(x-1) = 0 \rightarrow x = 0, x = 1$;

$$\text{Area} = \int_0^1 (\text{TOP} - \text{BOTTOM})\,dx$$

$$+ \int_1^2 (\text{TOP} - \text{BOTTOM})\,dx$$

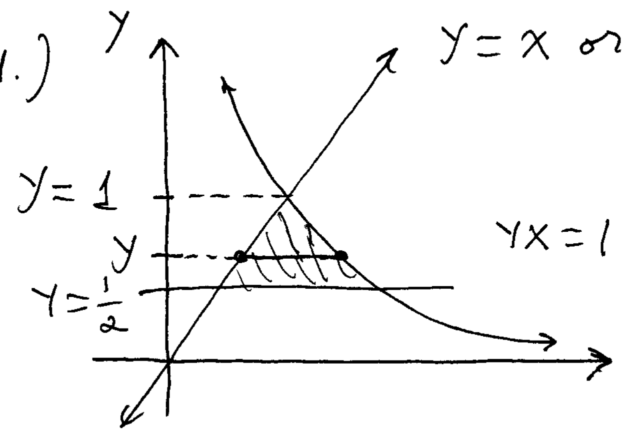
$$= \int_0^1 (x^2 - x^3)\,dx + \int_1^2 (x^3 - x^2)\,dx$$

$$= \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 + \left(\frac{1}{4}x^4 - \frac{1}{3}x^3\right)\Big|_1^2$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) - (0-0) + \left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)$$

$$= \frac{1}{3} - \frac{1}{4} + 4 - \frac{8}{3} - \frac{1}{4} + \frac{1}{3} = \frac{-1}{2} + 4 - \frac{6}{3} = \frac{3}{2}$$

14.)



$\left.\begin{array}{l} y = x \text{ or } x = y \\ yx = 1 \text{ or } x = 1/y \end{array}\right\} \quad y = \frac{1}{y} \rightarrow y^2 = 1 \rightarrow y = 1$

$$\text{Area} = \int_{\frac{1}{2}}^{1} (\text{RIGHT} - \text{LEFT})\,dy$$

$$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{y} - y\right) dy = \left(\ln y - \frac{1}{2}y^2\right)\Big|_{\frac{1}{2}}^{1}$$

$$= (\overset{0}{\ln 1} - \frac{1}{2}) - (\ln(1/2) - \frac{1}{8})$$

$$= -\frac{3}{8} - \ln(1/2)$$

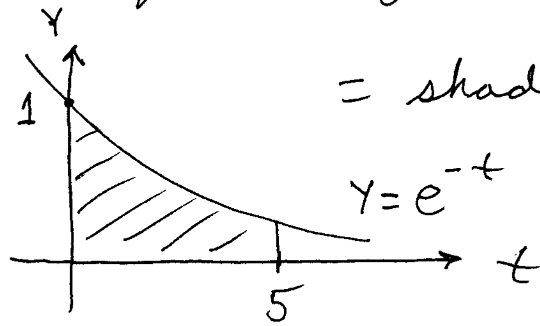
17.) a.) N: population, t: time

$\frac{dN}{dt} = e^{-t} \longrightarrow N = -e^{-t} + c$ and

$t=0,\ N=100$ so $100 = -e^0 + c \rightarrow$
$100 = -1 + c \rightarrow c = 101 \rightarrow \boxed{N = 101 - e^{-t}}$

b.) $N(5) - N(0) = (101 - e^{-5}) - (101 - 1)$
$= 1 - e^{-5}$ (cum. change)

c.) Change $= \int_0^5 \frac{dN}{dt}\,dt = \int_0^5 e^{-t}\,dt$



= shaded area

20.) $a(t) = \frac{d}{dt} v(t)$ so cum. change in $v(t)$ on $[0,T]$ is

Change $= \int_0^T \frac{d}{dt} v(t)\,dt$

$= \int_0^T a(t)\,dt = \int_0^T 32\,dt = 32t\Big|_0^T$

$= 32(T) - 32(0) = 32T$

22.) w: lbs., x: months then

$\int_3^5 \frac{dw}{dx}\,dx = w(x)\Big|_3^5 = w(5) - w(3)$

is cum. weight change for 3 to 5 months

24.) N: people, t: years

$$\text{Change} = \int_0^3 \frac{dN}{dt}\,dt = N(t)\Big|_0^3 = N(3) - N(0)$$

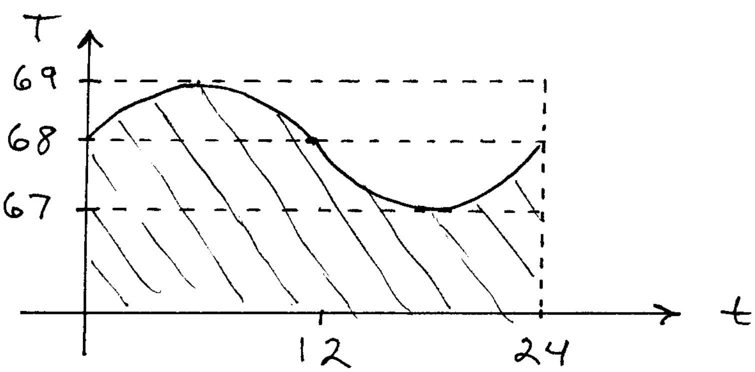
25.) $$AVE = \frac{1}{2-0}\int_0^2 (x^2 - 2)\,dx = \frac{1}{2}\left(\frac{1}{3}x^3 - 2x\right)\Big|_0^2$$

$$= \left(\frac{1}{6}x^3 - x\right)\Big|_0^2 = \frac{4}{3} - 2 = -\frac{2}{3}$$

26.) $$AVE = \frac{1}{1-(-1)}\int_{-1}^{1} \sin \pi t\,dt = \frac{1}{2}\cdot\frac{-1}{\pi}\cos \pi t\Big|_{-1}^{1}$$

$$= \frac{-1}{2\pi}\underbrace{\cos \pi}_{-1} - \frac{-1}{2\pi}\underbrace{\cos(-\pi)}_{-1} = \frac{1}{2\pi} - \frac{1}{2\pi} = 0$$

27.) a.)



b.) $$AVE = \frac{1}{24-0}\int_0^{24}\left(68 + \sin\left(\frac{\pi}{12}t\right)\right)dt$$

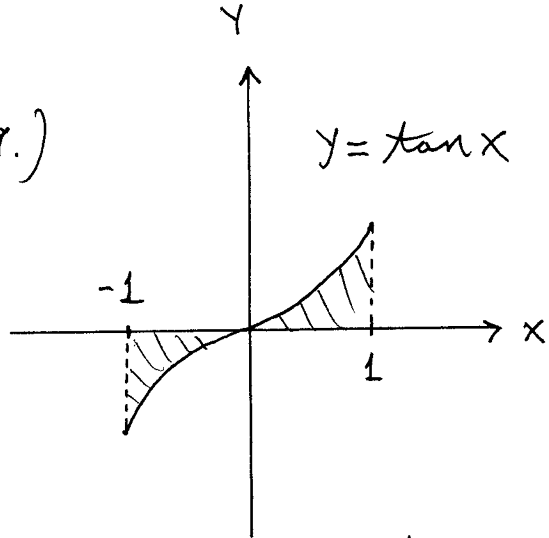
$$= \frac{1}{24}\left(68t - \frac{12}{\pi}\cos\left(\frac{\pi}{12}t\right)\right)\Big|_0^{24}$$

$$= \frac{1}{24}\left(68(24) - \frac{12}{\pi}\overbrace{\cos(2\pi)}^{1}\right)$$

$$- \frac{1}{24}\left(68(0) - \frac{12}{\pi}\overbrace{\cos 0}^{1}\right) = 68\,^\circ F$$

Note: Rectangle of base 24 and height 68 has same area as shaded region.

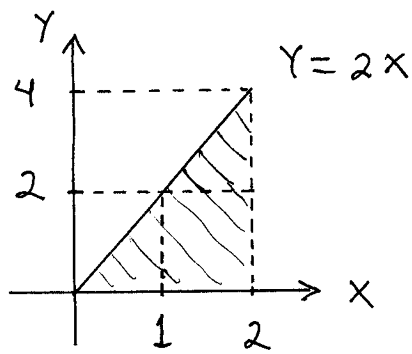
29.)



The area above the x-axis matches the area below the x-axis, so

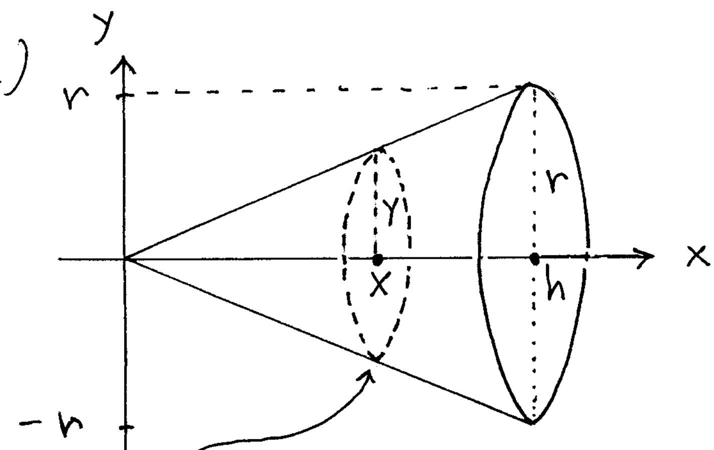
$$AVE = \frac{1}{1-(-1)} \int_{-1}^{1} \tan x \, dx = \frac{1}{2}(0) = 0$$

31.)



AVE = 2 and function equals this value at $x = 1$.

33.)



Slice at x is a circle

By similar Δ's

$$\frac{Y}{x} = \frac{r}{h} \rightarrow$$

$$Y = \frac{r}{h} x \; ;$$

Slice at x has area $A(x) = \pi (\text{radius})^2$

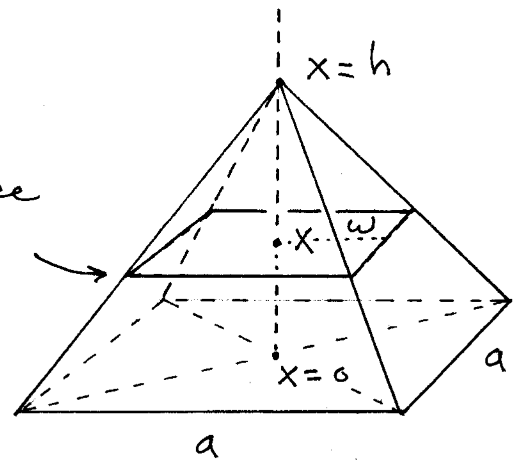
$$= \pi \left(\frac{r}{h} x\right)^2 \rightarrow \boxed{A(x) = \pi \frac{r^2}{h^2} x^2} \; ; \text{ then}$$

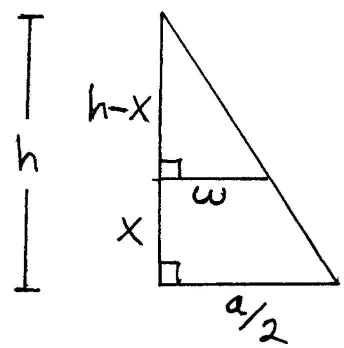
$$\text{Volume} = \int_0^h A(x) \, dx = \int_0^h \pi \frac{r^2}{h^2} x^2 \, dx$$

$$= \pi \frac{r^2}{h^2}\cdot\frac{1}{3}x^3 \Big|_0^h = \pi\frac{r^2}{h^2}\cdot\frac{1}{3}h^3 = \frac{1}{3}\pi r^2 h$$

34.)

Slice at x is a square





SIDE VIEW

By similar Δ's $\quad \dfrac{w}{h-x} = \dfrac{a/2}{h} = \dfrac{a}{2h} \rightarrow$

$w = \dfrac{a}{2h}(h-x)$; so slice at x has

area $A(x) = (2w)^2 = \left(\cancel{2}\cdot\dfrac{a}{\cancel{2}h}(h-x)\right)^2$

$= \dfrac{a^2}{h^2}(h^2-2hx+x^2) \rightarrow \boxed{A(x) = \dfrac{a^2}{h^2}(h^2-2hx+x^2)}$;

then $\text{Volume} = \int_0^h A(x)\,dx$

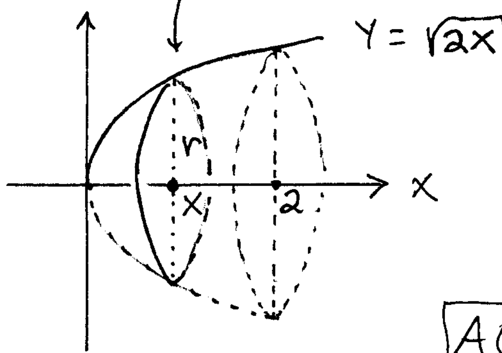
$= \int_0^h \dfrac{a^2}{h^2}(h^2-2hx+x^2)\,dx$

$= \dfrac{a^2}{h^2}\left(h^2x - hx^2 + \dfrac{1}{3}x^3\right)\Big|_0^h$

$= \dfrac{a^2}{h^2}\left(\cancel{h^3} - \cancel{h^3} + \dfrac{1}{3}h^3\right) = \dfrac{1}{3}a^2h$

Slice at x is a circle

36.)



Slice at x has area $A(x) = \pi r^2$

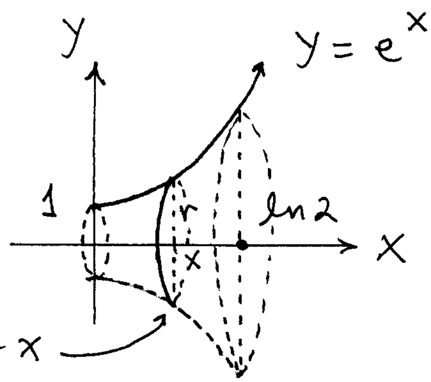
$= \pi (\sqrt{2x})^2 \longrightarrow$

$\boxed{A(x) = 2\pi x}$; then

$$\text{Volume} = \int_0^2 A(x)\,dx = \int_0^2 2\pi x\,dx$$

$$= 2\pi \cdot \frac{1}{2} x^2 \Big|_0^2 = 4\pi$$

38.)



Slice at x is a circle

Slice at x has area

$A(x) = \pi r^2 = \pi (e^x)^2 \longrightarrow$

$\boxed{A(x) = \pi \cdot e^{2x}}$; then

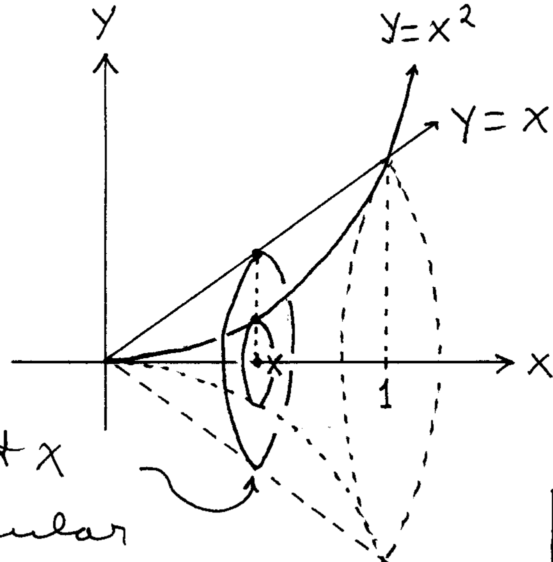
$$\text{Volume} = \int_0^{\ln 2} A(x)\,dx$$

$$= \int_0^{\ln 2} \pi \cdot e^{2x}\,dx = \pi \cdot \frac{1}{2} e^{2x} \Big|_0^{\ln 2}$$

$$= \frac{\pi}{2} e^{2\ln 2} - \frac{\pi}{2} e^0 = \frac{\pi}{2} e^{\ln 2^2} - \frac{\pi}{2}(1)$$

$$= \frac{\pi}{2}(4) - \frac{\pi}{2} = 3\frac{\pi}{2}$$

41.)



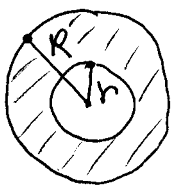
Slice at x has area

$$A(x) = \pi R^2 - \pi r^2$$

$$= \pi (x)^2 - \pi (x^2)^2 \rightarrow$$

$$\boxed{A(x) = \pi x^2 - \pi x^4} \ ;$$

Slice at x is a circular disc with a hole



~~then~~

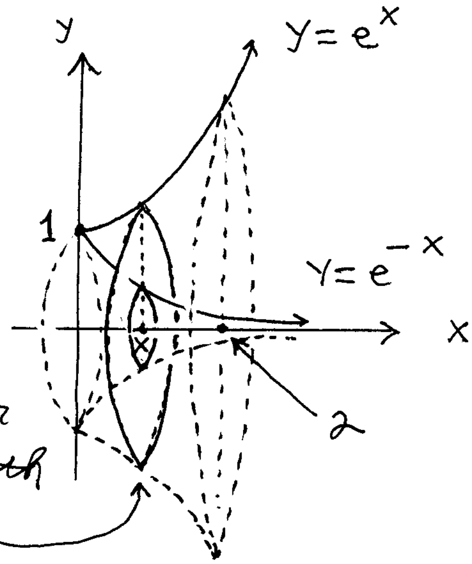
$$\text{Volume} = \int_0^1 A(x)\, dx$$

$$= \int_0^1 (\pi x^2 - \pi x^4)\, dx$$

$$= \left(\pi \cdot \frac{1}{3} x^3 - \pi \cdot \frac{1}{5} x^5\right)\Big|_0^1 = \frac{\pi}{3} - \frac{\pi}{5}$$

$$= \frac{5}{15}\pi - \frac{3}{15}\pi = \frac{2}{15}\pi$$

43.)



Slice at x has area

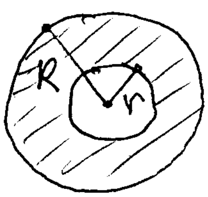
$$A(x) = \pi R^2 - \pi r^2$$

$$= \pi (e^x)^2 - \pi (e^{-x})^2$$

$$= \text{—} \longrightarrow$$

$$\boxed{A(x) = \pi e^{2x} - \pi e^{-2x}} \ ;$$

Slice at x is a circular disc with a hole



then

$$\text{Volume} = \int_0^2 A(x)\, dx$$

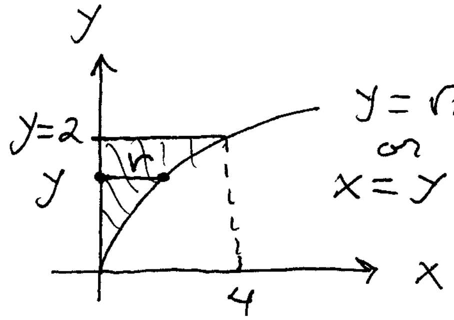
$$= \int_0^2 (\pi e^{2x} - \pi e^{-2x})\, dx$$

$$= \left(\pi \cdot \frac{1}{2} e^{2x} - \pi \cdot \frac{-1}{2} e^{-2x}\right)\Big|_0^2$$

$$= \left(\frac{\pi}{2} e^{4} + \frac{\pi}{2} e^{-4}\right) - \left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= \frac{\pi}{2} e^{4} + \frac{\pi}{2} e^{-4} - \pi$$

47.)



$y = \sqrt{x}$ or $x = y^2$

Slice at y has area

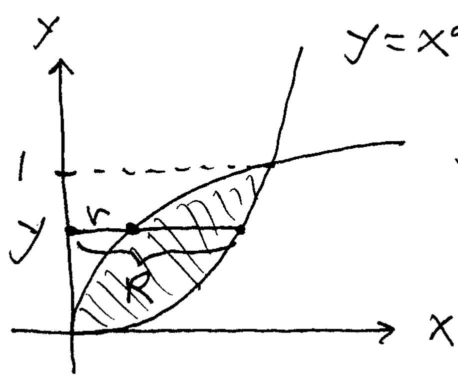
$A(y) = \pi r^2 = \pi (y^2)^2 \rightarrow$

$\boxed{A(y) = \pi y^4}$, then

$$\text{Volume} = \int_0^2 A(y)\, dy = \int_0^2 \pi y^4\, dy$$

$$= \pi \cdot \frac{1}{5} y^5 \Big|_0^2 = \frac{32}{5}\pi$$

51.)



$y = x^2$ or $x = \sqrt{y}$

$y = \sqrt{x}$ or $x = y^2$

Slice at y has area

$$A(y) = \pi R^2 - \pi r^2 = \pi (\sqrt{y})^2 - \pi (y^2)^2 \rightarrow$$

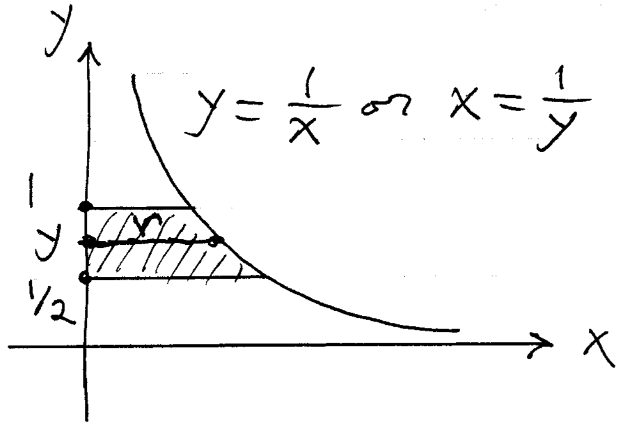
$\boxed{A(y) = \pi y - \pi y^4}$, then

$$\text{Volume} = \int_0^1 A(y)\, dy = \int_0^1 (\pi y - \pi y^4)\, dy$$

$$= \left(\pi \cdot \tfrac{1}{2}y^2 - \pi \cdot \tfrac{1}{5}y^5\right)\Big|_0^1 = \tfrac{1}{2}\pi - \tfrac{1}{5}\pi$$

$$= \tfrac{3}{10}\pi$$

52.)



$y = \frac{1}{x}$ or $x = \frac{1}{y}$

Slice at y has area

$A(y) = \pi r^2 = \pi\left(\frac{1}{y}\right)^2 \rightarrow$

$$\boxed{A(y) = \pi \cdot \frac{1}{y^2}},$$

~~then~~

$$\text{Volume} = \int_{1/2}^{1} A(y)\,dy = \int_{\frac{1}{2}}^{1} \pi \cdot \frac{1}{y^2}\,dy$$

$$= \pi \cdot \frac{-1}{y}\Big|_{1/2}^{1} = \pi(-1 - {}^-2) = \pi$$

53.) $Y = 2X$ on $[0,2]$:

$$\text{ARC} = \int_0^2 \sqrt{1 + (Y')^2}\,dx = \int_0^2 \sqrt{1 + (2)^2}\,dx$$

$$= \int_0^2 \sqrt{5}\,dx = \sqrt{5}\,x\Big|_0^2 = 2\sqrt{5}$$

55.) $Y^2 = X^3 \rightarrow Y = X^{3/2}$ on $[1,4]$:

$$\text{ARC} = \int_1^4 \sqrt{1 + \left(\tfrac{3}{2}X^{1/2}\right)^2}\,dx = \int_1^4 \sqrt{1 + \tfrac{9}{4}X}\,dx$$

$$= \frac{2}{3} \cdot \frac{4}{9} \cdot \left(1 + \tfrac{9}{4}X\right)^{3/2}\Big|_1^4$$

$$= \frac{8}{27}(10)^{3/2} - \frac{8}{27}\left(\frac{13}{4}\right)^{3/2}$$

57.) $y = \frac{1}{6}x^3 + \frac{1}{2}\cdot\frac{1}{x}$ on $[1, 3]$ $\xrightarrow{D}$

$y' = \frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{-1}{x^2} = \frac{1}{2}x^2 - \frac{1}{2x^2}$;

$$ARC = \int_1^3 \sqrt{1+(y')^2}\,dx = \int_1^3 \sqrt{1+\left(\frac{1}{2}x^2 - \frac{1}{2x^2}\right)^2}\,dx$$

$$= \int_1^3 \sqrt{1+\left(\frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}\right)}\,dx$$

$$= \int_1^3 \sqrt{\frac{1}{2} + \frac{x^4}{4} + \frac{1}{4x^4}}\,dx$$

$$= \int_1^3 \sqrt{\frac{2x^4}{4x^4} + \frac{x^8}{4x^4} + \frac{1}{4x^4}}\,dx$$

$$= \int_1^3 \frac{\sqrt{x^8 + 2x^4 + 1}}{\sqrt{4x^4}}\,dx = \int_1^3 \frac{\sqrt{(x^4+1)^2}\,dx}{2x^2}$$

$$= \int_1^3 \frac{x^4+1}{2x^2}\,dx = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)dx$$

$$= \left(\frac{1}{2}\cdot\frac{1}{3}x^3 + \frac{1}{2}\cdot\frac{-1}{x}\right)\Big|_1^3$$

$$= \left(\frac{9}{2} - \frac{1}{6}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{9}{2} - \frac{1}{6} - \frac{1}{6} + \frac{1}{2}$$

$$= 5 - \frac{1}{3} = \frac{14}{3}$$