Section 7.4

1.) $\int_0^\infty 3e^{-6x}\,dx = \lim_{A\to\infty} \int_0^A 3\cdot e^{-6x}\,dx$

$= \lim_{A\to\infty} 3\cdot \frac{-1}{6} e^{-6x}\Big|_0^A = \lim_{A\to\infty} \frac{-1}{2} e^{-6x}\Big|_0^A$ dx;

$= \lim_{A\to\infty}\left(\frac{-1}{2}e^{-6A} - \frac{-1}{2}\cancel{e^0}^{1}\right) = \lim_{A\to\infty}\left(\frac{-1}{2}\cdot\frac{1}{e^{6A}} + \frac{1}{2}\right)$

$= \frac{-1}{2}\cdot \cancel{\frac{1}{e^\infty}}^{0} + 1/2 = 1/2$

2.) $\int_0^\infty x e^{-x}\,dx = \lim_{A\to\infty}\int_0^A x e^{-x}\,dx$

(Let $u = x$, $dv = e^{-x}dx \to du = 1\,dx$, $v = -e^{-x}$)

$= \lim_{A\to\infty}\left[-xe^{-x}\Big|_0^A - -\int_0^A e^{-x}\,dx\right]$

$= \lim_{A\to\infty}\left[-Ae^{-A} - (0) + -e^{-x}\Big|_0^A\right]$

$= \lim_{A\to\infty}\left[\frac{-A}{e^A} - (e^{-A} - \cancel{e^0}^{1})\right]$

$= \lim_{A\to\infty}\left[\frac{-A}{e^A} - \frac{1}{e^A} + 1\right]$

"$\frac{\infty}{\infty}$" so use L'Hopital

$= \left(\lim_{A\to\infty}\frac{-1}{e^A}\right) - \frac{1}{e^\infty} + 1$

$= \frac{-1}{e^\infty} - \frac{1}{e^\infty} + 1 = \frac{-1}{\infty} - \frac{1}{\infty} + 1$

$= 0 - 0 + 1 = 1$

3.) $\int_0^\infty \frac{2}{1+x^2}\,dx = \lim_{A\to\infty} \int_0^A \frac{2}{1+x^2}\,dx$

$= \lim_{A\to\infty} 2\cdot \arctan x \Big|_0^A$

$= \lim_{A\to\infty} \left(2\arctan A - \cancelto{0}{2\arctan 0}\right)$

$= 2\cdot$ "$\arctan\infty$" $= 2\cdot\frac{\pi}{2} = \pi$

4.) $\int_e^\infty \frac{dx}{x(\ln x)^2} = \lim_{A\to\infty} \int_e^A \frac{dx}{x(\ln x)^2}$

(Let $u = \ln x \xrightarrow{D} du = \frac{1}{x}dx$; and $x: e \to A$ so $u: \ln e \to \ln A$, i.e., $u: 1 \to \ln A$)

$= \lim_{A\to\infty} \int_1^{\ln A} \frac{1}{u^2}\,du = \lim_{A\to\infty} \frac{-1}{u}\Big|_1^{\ln A}$

$= \lim_{A\to\infty} \left(\frac{-1}{\ln A} - \frac{-1}{1}\right) =$ "$\frac{-1}{\infty}$" $+ 1 = 0 + 1 = 1$

5.) $\int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{A\to\infty} \int_1^A x^{-3/2}\,dx$

$= \lim_{A\to\infty} \frac{x^{-1/2}}{-1/2}\Big|_1^A = \lim_{A\to\infty} -2\cdot\frac{1}{x^{1/2}}\Big|_1^A$

$= \lim_{A\to\infty} -2\cdot\frac{1}{\sqrt{A}} - -2\cdot\frac{1}{1} =$ "$-\frac{2}{\infty}$" $+ 2 = 0 + 2 = 2$

8.) $\int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}}\,dx = \int_{-\infty}^{0} x e^{-\frac{x^2}{2}}\,dx + \int_0^\infty x e^{-\frac{x^2}{2}}\,dx$

$= B + C$;

$B = \lim_{A\to-\infty} \int_A^0 x e^{-x^2/2}\,dx$ (Let $u = -\frac{x^2}{2} \xrightarrow{D}$

$du = -x\,dx \rightarrow -du = x\,dx$; and
$x: A \rightarrow 0$ so $u: -\frac{A^2}{2} \rightarrow 0$)

$$= \lim_{A\to -\infty} -\int_{-\frac{A^2}{2}}^{0} e^u\,du = \lim_{A\to-\infty} -e^u \Big|_{-\frac{A^2}{2}}^{0}$$

$$= \lim_{A\to-\infty} \left(-e^0 - -e^{-\frac{A^2}{2}}\right) = -(1) + (e^{-\infty})$$

$$= -1 + \cancelto{0}{\frac{1}{e^{\infty}}} = \boxed{-1} \; ;$$

$$C = \lim_{A\to\infty} \int_0^A x e^{-\frac{x^2}{2}} dx \quad (\text{Let } u = -\tfrac{x^2}{2} \xrightarrow{D}$$

$du = -x\,dx \rightarrow -du = x\,dx$; and
$x: 0 \rightarrow A$ so $u: 0 \rightarrow -\frac{A^2}{2}$)

$$= \lim_{A\to\infty} -\int_0^{-\frac{A^2}{2}} e^u\,du = \lim_{A\to\infty} -e^u \Big|_0^{-\frac{A^2}{2}}$$

$$= \lim_{A\to\infty} \left(-e^{-\frac{A^2}{2}} - -e^0\right) = -e^{-\infty} + (1)$$

$$= -\cancelto{0}{\frac{1}{e^{\infty}}} + 1 = \boxed{1} \; ; \text{ thus,}$$

$$\int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx = B + C = -1 + 1 = 0$$

9.) $$\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\,dx = \int_{-\infty}^{0} \frac{x}{(1+x^2)^2}dx + \int_0^{\infty} \frac{x}{(1+x^2)^2}dx$$

$$= B + C \; ;$$

$$B = \lim_{A\to-\infty} \int_A^0 \frac{x}{(1+x^2)^2}dx \quad (\text{Let } u = 1+x^2 \xrightarrow{D}$$

$du = 2x\,dx \rightarrow \frac{1}{2}du = x\,dx$; and
$x: A \rightarrow 0$ so $u: 1+A^2 \rightarrow 1$)

$$= \lim_{A\to-\infty} \frac{1}{2}\int_{1+A^2}^{1} \frac{1}{u^2}\,du = \lim_{A\to-\infty} \frac{1}{2}\cdot\frac{-1}{u}\Big|_{1+A^2}^{1}$$

$$= \lim_{A \to -\infty} \left(\frac{1}{2} \cdot \frac{-1}{1} - \frac{1}{2} \cdot \frac{-1}{1+A^2} \right)$$

$$= \frac{-1}{2} + \frac{1}{2} \cdot \cancel{\frac{1}{\infty}}^{\,0} = \boxed{\frac{-1}{2}} ;$$

$$C = \lim_{A \to \infty} \int_0^A \frac{x}{(1+x^2)^2}\, dx \quad (\text{Let } u = 1+x^2 \xrightarrow{D}$$

$du = 2x\, dx \to \frac{1}{2} du = x\, dx$; and

$x : 0 \to A$ so $u : 1 \to 1+A^2$)

$$= \lim_{A \to \infty} \frac{1}{2} \int_1^{1+A^2} \frac{1}{u^2}\, du = \lim_{A \to \infty} \frac{1}{2} \cdot \frac{-1}{u} \Big|_1^{1+A^2}$$

$$= \lim_{A \to \infty} \left(\frac{1}{2} \cdot \frac{-1}{1+A^2} - \frac{1}{2} \cdot \frac{-1}{1} \right)$$

$$= \frac{1}{2} \cdot \cancel{\frac{-1}{\infty}}^{\,0} + \frac{1}{2} = \boxed{\frac{1}{2}} ; \text{ thus,}$$

$$\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\, dx = B + C = \frac{-1}{2} + \frac{1}{2} = 0 .$$

11.) $$\int_0^9 \frac{dx}{\sqrt{9-x}} = \lim_{A \to 9^-} \int_0^A \frac{dx}{\sqrt{9-x}} \quad (\text{Let } u = 9-x$$

$\xrightarrow{D} du = -dx \to -du = dx$; and

$x : 0 \to A$ so $u : 9 \to 9-A$)

$$= \lim_{A \to 9^-} -\int_9^{9-A} \frac{1}{u^{1/2}}\, du = \lim_{A \to 9^-} -\int_9^{9-A} u^{-1/2}\, du$$

$$= \lim_{A \to 9^-} -\frac{u^{1/2}}{1/2} \Big|_9^{9-A} = \lim_{A \to 9^-} -2\sqrt{u} \Big|_9^{9-A}$$

$$= \lim_{A \to 9^-} \left(-2\sqrt{9-A} - -2\sqrt{9} \right)$$

$$= -2(0) + 2(3) = 6$$

14.) $\int_{-2}^{0} \frac{dx}{(x+1)^{1/3}} = \int_{-2}^{-1} \frac{dx}{(x+1)^{1/3}} + \int_{-1}^{0} \frac{dx}{(x+1)^{1/3}}$

$= \quad B \quad + \quad C \quad ;$

$B = \lim_{A \to -1^-} \int_{-2}^{A} \frac{dx}{(x+1)^{1/3}}$ (Let $u = x+1 \xrightarrow{D} du = dx$;

$x: -2 \to A$ so $u: -1 \to A+1$)

$= \lim_{A \to -1^-} \int_{-1}^{A+1} \frac{1}{u^{1/3}}\, du = \lim_{A \to -1^-} \int_{-1}^{A+1} u^{-1/3}\, du$

$= \lim_{A \to -1^-} \frac{u^{2/3}}{2/3} \Big|_{-1}^{A+1} = \lim_{A \to -1^-} \left(\frac{3}{2}(A+1)^{2/3} - \frac{3}{2}(-1)^{2/3} \right)$

$= \frac{3}{2}(0)^{2/3} - \frac{3}{2}(1) = \boxed{-\frac{3}{2}}$;

$C = \lim_{A \to -1^+} \int_{A}^{0} \frac{dx}{(x+1)^{1/3}}$ (Let $u = x+1 \xrightarrow{D} du = dx$;

$x: A \to 0$ so $u: A+1 \to 1$)

$= \lim_{A \to -1^+} \int_{A+1}^{1} \frac{1}{u^{1/3}}\, du = \lim_{A \to -1^+} \frac{u^{2/3}}{2/3} \Big|_{A+1}^{1}$

$= \lim_{A \to -1^+} \left(\frac{3}{2}(1)^{2/3} - \frac{3}{2}(A+1)^{2/3} \right)$

$= \frac{3}{2}(1) - \frac{3}{2}(0) = \boxed{\frac{3}{2}}$; thus,

$\int_{-2}^{0} \frac{dx}{(x+1)^{1/3}} = B + C = -\frac{3}{2} + \frac{3}{2} = 0$.

17.) $\int_{1}^{\infty} \frac{1}{x^3}\, dx = \lim_{A \to \infty} \int_{1}^{A} x^{-3}\, dx$

$= \lim_{A \to \infty} \frac{x^{-2}}{-2} \Big|_{1}^{A} = \lim_{A \to \infty} \left(\frac{-1}{2} \cdot \frac{1}{x^2} \right) \Big|_{1}^{A}$

$$= \lim_{A\to\infty} \left(\frac{-1}{2}\cdot \overset{0}{\cancel{\frac{1}{A^2}}} - \frac{-1}{2}\cdot\frac{1}{1^2} \right) = -\frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$$

18.) $\int_1^\infty \frac{1}{x^{1/3}}\,dx = \lim_{A\to\infty} \int_1^A x^{-1/3}\,dx$

$$= \lim_{A\to\infty} \frac{x^{2/3}}{2/3}\Big|_1^A = \lim_{A\to\infty} \left(\frac{3}{2}\cdot A^{2/3} - \frac{3}{2}(1)^{2/3} \right)$$

$$= \frac{3}{2}(\infty) - \frac{3}{2}(1) = \infty - \frac{3}{2} = \infty$$

20.) $\int_0^4 \frac{1}{x^{1/4}}\,dx = \lim_{A\to 0^+} \int_A^4 x^{-1/4}\,dx$

$$= \lim_{A\to 0^+} \frac{x^{3/4}}{3/4}\Big|_A^4 = \lim_{A\to 0^+} \left(\frac{4}{3}(4)^{3/4} - \frac{4}{3} A^{3/4} \right)$$

$$= \frac{1}{3}\cdot 4^{7/4} - \frac{4}{3}(0)^{3/4} = \frac{1}{3}\cdot 4^{7/4}$$

25.) $\int_e^\infty \frac{dx}{x\ln x} = \lim_{A\to\infty} \int_e^A \frac{dx}{x\ln x}$ (Let $u = \ln x$

$\xrightarrow{D} du = \frac{1}{x}\,dx$; $x: e \to A$ so $u: \ln e \to \ln A$, i.e.,

$u: 1 \to \ln A$)

$$= \lim_{A\to\infty} \int_1^{\ln A} \frac{1}{u}\,du = \lim_{A\to\infty} \ln|u| \Big|_1^{\ln A}$$

$$= \lim_{A\to\infty} \left(\ln|\ln A| - \overset{0}{\cancel{\ln 1}} \right) = \text{"}\ln(\infty)\text{"} = \infty.$$

28.) $\int_{-\infty}^1 \frac{3}{1+x^2}\,dx = \lim_{A\to-\infty} \int_A^1 \frac{3}{1+x^2}\,dx$

$$= \lim_{A\to-\infty} 3\cdot \arctan x \Big|_A^1$$

$$= \lim_{A\to-\infty} (3\arctan 1 - 3\arctan A)$$

$$= 3\cdot\left(\frac{\pi}{4}\right) - 3\cdot\arctan(-\infty) = 3\left(\frac{\pi}{4}\right) - 3\left(-\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{4} + \frac{3\pi}{2} = \frac{9\pi}{4}$$

29.) $\int_0^{\infty} \frac{1}{x^2-1}\,dx = \int_0^1 \frac{1}{x^2-1}\,dx + \int_1^{\infty} \frac{1}{x^2-1}\,dx$

$$= \int_0^1 \frac{1}{x^2-1}\,dx + \int_1^2 \frac{1}{x^2-1}\,dx + \int_2^{\infty} \frac{1}{x^2-1}\,dx$$

$$= \quad B \quad + \quad C \quad + \quad D \; ;$$

$$B = \lim_{A\to 1^-} \int_0^A \frac{1}{x^2-1}\,dx = \lim_{A\to 1^-} \int_0^A \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx$$

$$= \lim_{A\to 1^-} \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right)\Big|_0^A$$

$$= \lim_{A\to 1^-} \left\{\frac{1}{2}\left(\ln|A-1| - \ln|A+1|\right) - \frac{1}{2}\left(\overset{0}{\ln 1} - \overset{0}{\ln 1}\right)\right\}$$

$$= \frac{1}{2}\left(\ln(0^+) - \ln 2\right) = \frac{1}{2}\left(-\infty - \ln 2\right) = -\infty ,$$

so $\int_0^{\infty} \frac{1}{x^2-1}\,dx$ diverges.

31.) $\int_0^{\infty} c e^{-3x}\,dx = 1 \rightarrow$

$$\lim_{A\to\infty} \int_0^A c e^{-3x}\,dx = 1 \rightarrow$$

$$\lim_{A\to\infty} c \cdot \frac{e^{-3x}}{-3}\Big|_0^A = 1 \rightarrow$$

$$\lim_{A\to\infty} -\frac{c}{3}\left(e^{-3A} - e^0\right) = -\frac{c}{3}\left(\overset{0}{\cancel{\frac{1}{e^{\infty}}}} - 1\right) = 1 \rightarrow$$

$$\frac{c}{3} = 1 \rightarrow c = 3$$

32.) $\displaystyle\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = 1 \rightarrow$

$$\int_{-\infty}^{0} \frac{c}{1+x^2}\,dx + \int_{0}^{\infty} \frac{c}{1+x^2}\,dx = 1 \rightarrow$$

$$B \quad + \quad C \quad = 1 \;;$$

$$B = \lim_{A\to-\infty} \int_A^0 \frac{c}{1+x^2}\,dx$$

$$= \lim_{A\to-\infty} c\cdot \arctan x \Big|_A^0$$

$$= \lim_{A\to-\infty} \left(c\cdot \cancelto{0}{\arctan 0} - c \arctan A\right)$$

$$= c\cdot(0) - c\cdot\left(-\frac{\pi}{2}\right) = c\frac{\pi}{2} \;;$$

$$C = \lim_{A\to\infty} \int_0^A \frac{c}{1+x^2}\,dx$$

$$= \lim_{A\to\infty} c\cdot \arctan x \Big|_0^A$$

$$= \lim_{A\to\infty} \left(c\cdot \arctan A - c\cdot \cancelto{0}{\arctan 0}\right)$$

$$= c\cdot\left(\frac{\pi}{2}\right) - c(0) = c\frac{\pi}{2}\;;$$ ~~then~~

$$c\frac{\pi}{2} + c\frac{\pi}{2} = 1 \rightarrow c\pi = 1 \rightarrow c = \frac{1}{\pi}$$