Math 17B
Kouba
Inverses and Determinants of Matrices

DEFINITION : Let A be an $n\ x\ n$ matrix. Matrix A^{-1} is the *inverse* of matrix A if

$$AA^{-1} = A^{-1}A = I_n \text{ , the } n\ x\ n \text{ identity matrix.}$$

We say that matrix A is *invertible* .

EXAMPLE 1: Let $A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$. Consider matrix $\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$. Then

$$\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

so that A is invertible and $A^{-1} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$.

EXAMPLE 2: Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$. Consider matrix $\begin{pmatrix} -1 & 0 & 1 \\ 5/3 & 1/3 & -4/3 \\ -1/3 & -2/3 & 2/3 \end{pmatrix}$. Then

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 5/3 & 1/3 & -4/3 \\ -1/3 & -2/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 0 & 1 \\ 5/3 & 1/3 & -4/3 \\ -1/3 & -2/3 & 2/3 \end{pmatrix}\begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

so that A is invertible and $A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 5/3 & 1/3 & -4/3 \\ -1/3 & -2/3 & 2/3 \end{pmatrix}$.

HOW TO FIND INVERSES : To find A^{-1} for matrix A :

1.) Form matrix $[A : I_n]$.
2.) Use matrix reduction rules to create matrix $[I_n : B]$.
3.) Then $B = A^{-1}$.

NOTE : Not all $n\ x\ n$ matrices have inverses.

EXAMPLE 3: Find the inverse of each matrix.

1.) $A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} \longrightarrow \left(\begin{array}{cc|cc} 5 & 7 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & -2 \\ 2 & 3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{array}\right)$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 3 & -7 \\ 0 & 1 & -2 & 5 \end{array}\right), \text{ so that } A^{-1} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}.$$

2.) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 0 & 1/2 \\ 0 & 1 & 0 & -3 & 1 & -1 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right), \text{ so that } A^{-1} = \begin{pmatrix} 3/2 & 0 & 1/2 \\ -3 & 1 & -1 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

DETERMINANTS for 2 *x* 2 MATRICES

DEFINITION : Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The *determinant* of matrix A is the number given by

$$det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

EXAMPLE 4: $det \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} = (1)(4) - (-1)(3) = 4 + 3 = 7$.

EXAMPLE 5: $det \begin{pmatrix} -1 & -2 \\ 4 & 8 \end{pmatrix} = (-1)(8) - (-2)(4) = -8 + 8 = 0$.

THEOREM : Matrix A is invertible (nonsingular) if and only if $detA = 0$.

EXAMPLE 6: Matrix A in EXAMPLE 4 is invertible since $detA \neq 0$. Matrix A in EXAMPLE 5 is NOT invertible since $detA = 0$.