Lecture 1

Math 17 B
Laplace Transforms 1
HW #18

Laplace Transforms

These will be used to solve D.E.'s and systems of D.E.'s

Recall (From Math 17A and 17B):

1.) (Property of Integral)

$$\int_a^b (\alpha f(t) + \beta g(t))\,dt = \alpha \int_a^b f(t)\,dt + \beta \int_a^b g(t)\,dt$$

2.) (Improper Integral)

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty} \int_a^A f(t)\,dt$$

3.) (Integration by Parts)

$$\int u\,dv = uv - \int v\,du$$

4.) (Formula) $\int e^{kt}\,dt = \frac{1}{k}e^{kt} + c$

5.) (Squeeze Principle)

Ex: $-1 \le \cos x \le +1 \longrightarrow$

$$\frac{-1}{e^x} \leq \frac{\cos x}{e^x} \leq \frac{+1}{e^x}, \text{ then}$$

$$\lim_{x\to\infty} \frac{-1}{e^x} = \text{"}\frac{-1}{\infty}\text{"} = 0 \text{ and}$$

$$\lim_{x\to\infty} \frac{1}{e^x} = \text{"}\frac{1}{\infty}\text{"} = 0, \text{ so}$$

$$\lim_{x\to\infty} \frac{\cos x}{e^x} = 0;$$

<u>Ex</u>: Similarly, $\displaystyle\lim_{x\to\infty} \frac{\sin x}{e^x} = 0$.

6.) (L'Hopital's Rule)

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \text{"}\frac{\infty}{\infty}\text{"} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

<u>Def</u>: Let $Y = f(t)$ be defined for $t \geq 0$. Then the improper integral

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt$$

is called the <u>Laplace transform</u> of $Y = f(t)$.

Notation : $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) = F(s)$.

Ex : Determine the Laplace transform for each function .

1.) $f(t) = 2 \longrightarrow \mathcal{L}\{2\} = \int_0^\infty e^{-st} \cdot (2)\, dt$

$= \lim_{A \to \infty} 2 \int_0^A e^{-st}\, dt$ (assume $s > 0$.)

(Recall : $\int e^{kt}\, dt = \frac{1}{k} e^{kt} + c$)

$= \lim_{A \to \infty} 2 \cdot \frac{-1}{s} e^{-st} \Big|_0^A$

$= \lim_{A \to \infty} -\frac{2}{s}\left(e^{-sA} - e^{0}\right)$

$= \lim_{A \to \infty} \frac{-2}{s}\left(\frac{1}{e^{sA}} - 1\right)$

$= \frac{-2}{s}\left(\overset{0}{\cancel{\frac{1}{\infty}}} - 1\right) = \frac{2}{s}$

, i.e.,

$\mathcal{L}\{2\} = \frac{2}{s}$ for $s > 0$

2.) $f(t) = t+1 \longrightarrow \mathcal{L}\{t+1\} = \int_0^\infty e^{-st}(t+1)\, dt$

$= \lim_{A \to \infty} \int_0^A (t+1) e^{-st}\, dt$ (assume $s > 0$.)

(Let $u = t+1$, $dv = e^{-st}\, dt$

$du = 1\, dt$, $v = \frac{-1}{s} e^{-st}$)

$$= \lim_{A\to\infty}\left[\frac{-1}{s}e^{-st}(t+1)\Big|_0^A - \frac{-1}{s}\int_0^A e^{-st}\,dt\right]$$

$$= \lim_{A\to\infty}\left[\frac{-1}{s}\cdot\frac{(A+1)}{e^{sA}} - \frac{-1}{s} + \frac{1}{s}\cdot\frac{-1}{s}e^{-st}\Big|_0^A\right]$$

$$= \lim_{A\to\infty}\left[\frac{-1}{s}\cdot\underbrace{\frac{A+1}{e^{sA}}}_{\text{"}\frac{\infty}{\infty}\text{"}} + \frac{1}{s} - \frac{1}{s^2}\left(\frac{1}{e^{sA}}-1\right)\right]$$

so use L'Hopital

(<u>Recall</u>: $\lim_{x\to\infty}\frac{f(x)}{g(x)} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$)

$$= \frac{-1}{s}\cdot\lim_{A\to\infty}\frac{1}{se^{sA}} + \frac{1}{s} - \frac{1}{s^2}\left(\text{"}\frac{1}{\infty}\text{"}-1\right)$$

$$= \frac{-1}{s}\cdot\text{"}\frac{1}{\infty}\text{"} + \frac{1}{s} - \frac{1}{s^2}(0-1)$$

$$= \frac{-1}{s}\cdot(0) + \frac{1}{s} + \frac{1}{s^2} \qquad (\text{for } s>0)\text{, i.e.,}$$

$$\mathcal{L}\{t+1\} = \frac{1}{s^2} + \frac{1}{s} \qquad \text{for } s>0.$$

3.) $f(t) = e^{3t} \longrightarrow \mathcal{L}\{e^{3t}\} = \int_0^\infty e^{-st}\cdot e^{3t}\,dt$

$$= \lim_{A\to\infty}\int_0^A e^{3t-st}\,dt$$

$$= \lim_{A\to\infty}\int_0^A e^{(3-s)t}\,dt \qquad (\text{assume } s>3.)$$

$$= \lim_{A\to\infty}\frac{1}{3-s}e^{(3-s)t}\Big|_0^A$$

$$= \lim_{A\to\infty}\left[\frac{1}{3-s}e^{(3-s)A} - \frac{1}{3-s}\right]$$

$$= \lim_{A\to\infty} \left[\frac{1}{3-s} \cdot \frac{1}{e^{(s-3)A}} + \frac{1}{s-3} \right]$$

$$= \frac{1}{3-s} \cdot \frac{\text{“}1\text{”}}{\infty} + \frac{1}{s-3} = \frac{1}{3-s}(0) + \frac{1}{s-3} \text{, i.e.,}$$

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3} \text{ for } s > 3$$

4.) $f(t) = \sin t \rightarrow \mathcal{L}\{\sin t\} = \int_0^\infty e^{-st} \cdot \sin t \, dt$

$$= \lim_{A\to\infty} \underbrace{\int_0^A e^{-st} \sin t \, dt}_{\text{call it } K} \quad (\text{assume } s>0.)$$

call it K : (Let $u = e^{-st}$, $dv = \sin t\, dt$

$\rightarrow \quad du = -se^{-st} dt$, $v = -\cos t$)

$$K = \int_0^A e^{-st} \sin t \, dt = -\cos t \, e^{-st} \Big|_0^A - s\int_0^A e^{-st} \cos t \, dt$$

$$= \left(-\frac{\cos A}{e^{sA}} - -1 \right) - s\int_0^A e^{-st} \cos t \, dt$$

(Let $u = e^{-st}$, $dv = \cos t \, dt$

$\rightarrow \quad du = -se^{-st}$, $v = \sin t$)

$$= \left(-\frac{\cos A}{e^{sA}} + 1 \right) - s\left[e^{-st} \sin t \Big|_0^A - -s\int_0^A e^{-st} \sin t \, dt \right]$$

$$= \frac{-\cos A}{e^{sA}} + 1 - s\left[\frac{\sin A}{e^{sA}} - 0 + s \cdot \underbrace{\int_0^A e^{-st} \sin t \, dt}_{K} \right] \rightarrow$$

$$K = \frac{-\cos A}{e^{sA}} + 1 - s \cdot \frac{\sin A}{e^{sA}} - s^2 K \quad \rightarrow$$

$$K + s^2 K = -\frac{\cos A}{e^{sA}} + 1 - s \cdot \frac{\sin A}{e^{sA}} \longrightarrow$$

$$(1+s^2)K = -\frac{\cos A}{e^{sA}} + 1 - s \cdot \frac{\sin A}{e^{sA}} \longrightarrow$$

$$K = \frac{1}{1+s^2}\left[-\frac{\cos A}{e^{sA}} + 1 - s \cdot \frac{\sin A}{e^{sA}}\right];$$

then

$$\mathcal{L}\{\sin t\} = \lim_{A\to\infty} K$$

By Squeeze Principle

$$= \lim_{A\to\infty} \frac{1}{1+s^2}\left[\underbrace{-\cancel{\frac{\cos A}{e^{sA}}}}_{\to 0} + 1 - s \cdot \underbrace{\cancel{\frac{\sin A}{e^{sA}}}}_{\to 0}\right]$$

$$= \frac{1}{1+s^2}[1] \longrightarrow \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

for $s > 0$

5.) $\mathcal{L}\{te^{2t}\} = \lim_{A\to\infty} \int_0^A e^{-st} \cdot te^{2t}\,dt$

$= \lim_{A\to\infty} \int_0^A t \cdot e^{(2-s)t}\,dt$ (assume $s>2$.)

(Let $u=t$, $dv = e^{(2-s)t}dt$

$\rightarrow$ $du = dt$, $v = \frac{1}{2-s} e^{(2-s)t}$)

$= \lim_{A\to\infty} \left[\frac{t}{2-s} e^{(2-s)t} \Big|_0^A - \frac{1}{2-s} \int_0^A e^{(2-s)t}\,dt \right]$

$= \lim_{A\to\infty} \left[\frac{1}{2-s} \cdot \frac{A}{e^{(s-2)A}} - \frac{1}{2-s} \cdot \frac{1}{2-s} e^{(2-s)t} \Big|_0^A \right]$

"$\frac{\infty}{\infty}$" so use L'Hopital

$= \lim_{A\to\infty} \left[\frac{1}{2-s} \cdot \frac{1}{(s-2)e^{(s-2)A}} - \frac{1}{(2-s)^2} \left(\frac{1}{e^{(s-2)A}} - 1 \right) \right]$

"$\frac{1}{\infty}$" $= 0$ "$\frac{1}{\infty}$" $= 0$

$= \frac{1}{2-s} \cdot (0) - \frac{1}{(s-2)^2} \cdot (-1) = \frac{1}{(s-2)^2}$

for $s>2$