Lecture 2

Math 17B
Laplace Transforms 2
Hw # 19

<u>Laplace Transforms</u> (cont'd.)

<u>Recall</u>: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ so

1.) $\mathcal{L}\{\alpha f(t)\} = \int_0^\infty e^{-st} \cdot \alpha f(t)\,dt$
$= \alpha \int_0^\infty e^{-st} f(t)\,dt = \alpha \mathcal{L}\{f(t)\}$;

2.) $\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty e^{-st}(f(t)+g(t))\,dt$
$= \int_0^\infty (e^{-st} \cdot f(t) + e^{-st} \cdot g(t))\,dt$
$= \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} g(t)\,dt$
$= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$.

<u>Ex</u>: Use known formulas to find the Laplace transform of each function.

1.) $\mathcal{L}\{4+t\} = \mathcal{L}\{4 \cdot 1\} + \mathcal{L}\{t\}$
$= 4\mathcal{L}\{1\} + \mathcal{L}\{t\} = 4 \cdot \frac{1}{s} + \frac{1}{s^2}$

2.) $\mathcal{L}\{2e^{t} - 3e^{-2t}\}$

$= 2\mathcal{L}\{e^{t}\} - 3\mathcal{L}\{e^{-2t}\}$

$= 2 \cdot \dfrac{1}{s-1} - 3 \cdot \dfrac{1}{s-(-2)}$

$= \dfrac{2}{s-1} - \dfrac{3}{s+2}$

3.) $\mathcal{L}\{3 \cdot \sin 4t + \cos t\}$

$= 3\mathcal{L}\{\sin 4t\} + \mathcal{L}\{\cos t\}$

$= 3 \cdot \dfrac{4}{s^2+4^2} + \dfrac{s}{s^2+1^2}$

4.) $\mathcal{L}\{\sin t + e^{3t}\cos 4t\}$

$= \mathcal{L}\{\sin t\} + \mathcal{L}\{e^{3t}\cos 4t\}$

$= \dfrac{1}{s^2+1^2} + \dfrac{s-3}{(s-3)^2+4^2}$

Inverse Laplace Transforms

Def : If $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$, then $\mathcal{L}^{-1}\{F(s)\} = f(t)$ is the inverse Laplace transform of $F(s)$.

Ex : 1.) $\mathcal{L}\{1\} = \frac{1}{s}$ so $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$

2.) $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+3^2}$ so

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \sin 3t$$

3.) $\mathcal{L}\{t^3\} = \frac{3!}{t^4}$ so

$$\mathcal{L}^{-1}\left\{\frac{3!}{t^4}\right\} = t^3$$

4.) $\mathcal{L}\{e^{2t}\cos t\} = \frac{s-2}{(s-2)^2+1^2}$ so

$$\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1^2}\right\} = e^{2t}\cos t$$

Ex: Find the inverse Laplace transform for each $F(s)$.

1.) $F(s) = \frac{1}{s-5} \rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} = e^{5t}$

2.) $F(s) = \frac{5!}{s^6} \rightarrow \mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\} = t^5$

3.) $F(s) = \frac{s}{s^2+25} \rightarrow$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\} = \cos 5t$

4.) $F(s) = \frac{1}{s^2} + \frac{1}{s^2+16} \rightarrow$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{1}{4}\cdot\frac{4}{s^2+4^2}\right\} = t + \frac{1}{4}\sin 4t$

5.) $F(s) = \frac{1}{(s+4)^2} \rightarrow$

$\mathcal{L}^{-1}\left\{\frac{1}{(s-(-4))^2}\right\} = te^{-4t}$

6.) $F(s) = \frac{1}{(s-1)^2+9} \rightarrow$

$\mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{3}{(s-1)^2+3^2}\right\} = \frac{1}{3}e^t \sin 3t$

7.) $F(s) = \dfrac{5}{s^2 - s - 6} = \dfrac{5}{(s-3)(s+2)}$

$= \dfrac{A}{s-3} + \dfrac{B}{s+2} \longrightarrow$

$A(s+2) + B(s-3) = 5$:

<u>Let $s=3$</u> : $5A = 5 \rightarrow A = 1$

<u>Let $s=-2$</u> : $-5B = 5 \rightarrow B = -1$; then

$\mathcal{L}^{-1}\left\{\dfrac{5}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s-3} + \dfrac{-1}{s+2}\right\}$

$= e^{3t} - e^{-2t}$

8.) $F(s) = \dfrac{s+3}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+1} \longrightarrow$

$As(s^2+1) + B(s^2+1) + s^2(Cs+D) = s+3$:

<u>Let $s=0$</u> : $B = 3$

<u>Let $s=i$</u> : $A(0) + B(0) + (-1)(Ci+D) = i+3 \rightarrow$

$(-C)i + (-D) = (1)i + (3) \longrightarrow$

$-C = 1 \rightarrow C = -1$ and $-D = 3 \rightarrow$

$D = -3$

<u>Let $s=1$</u> : $2A + 3(2) + (-1+-3) = 4 \longrightarrow$

$2A + 6 - 4 = 4 \rightarrow 2A = 2 \rightarrow A = 1$;

then

$\mathcal{L}^{-1}\left\{\dfrac{s+3}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s} + \dfrac{3}{s^2} + \dfrac{-s-3}{s^2+1}\right\}$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s} + 3\cdot\frac{1}{s^2} - \frac{s}{s^2+1} - 3\cdot\frac{1}{s^2+1} \right\}$$

$$= 1 + 3t - \cos t - 3\sin t$$

9.) $$F(s) = \frac{s}{s^2+4s+13} = \frac{s}{(s^2+2s+4)+9}$$

$$= \frac{s}{(s+2)^2+3^2} = \frac{s+2-2}{(s+2)^2+3^2}$$

$$= \frac{s+2}{(s+2)^2+3^2} - 2\cdot\frac{1}{3}\cdot\frac{3}{(s+2)^2+3^2} \longrightarrow$$

$$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+2s+13} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s+2}{(s+2)^2+3^2} - \frac{2}{3}\cdot\frac{3}{(s+2)^2+3^2} \right\}$$

$$= e^{-2t}\cdot\cos 3t - \frac{2}{3}\cdot e^{-2t}\sin 3t$$