Math 17B
Laplace Transforms 3
HW # 20

Laplace Transforms applied to Derivatives

Def: Let $y = f(t)$ be defined for $t \geq 0$. We say f is of exponential order if there exist numbers c, $M > 0$, and $T > 0$ so that

$$|f(t)| \leq Me^{ct} \quad \text{for } t > T.$$

Question: For what functions does the Laplace transform exist?

Theorem: Let $y = f(t)$ be defined for $t \geq 0$. If f is

i.) continuous

and ii.) of exponential order,

then

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \quad \text{exists.}$$

Note: If f is of exponential order, i.e., $|f(t)| \leq Me^{ct}$ for $t > T$, then $-Me^{ct} \le f(t) \le Me^{ct} \to$

$$-Me^{cA} \le f(A) \le Me^{cA} \to$$

$$\frac{-Me^{cA}}{e^{sA}} \le \frac{f(A)}{e^{sA}} \le \frac{Me^{cA}}{e^{sA}} \to$$

$$-Me^{cA-sA} \le \frac{f(A)}{e^{sA}} \le Me^{cA-sA} \to$$

$$-Me^{(c-s)A} \le \frac{f(A)}{e^{sA}} \le Me^{(c-s)A} \to$$

$$\frac{-M}{e^{(s-c)A}} \le \frac{f(A)}{e^{sA}} \le \frac{M}{e^{(s-c)A}} \quad (\text{assume } s > c.);$$

then

$$\lim_{A\to\infty} \frac{-M}{e^{(s-c)A}} = 0 = \lim_{A\to\infty} \frac{M}{e^{(s-c)A}},$$

so by Squeeze Principle

$$\lim_{A\to\infty} \frac{f(A)}{e^{sA}} = 0 .$$

<u>Rule 1</u>: $\mathcal{L}\{Y'\} = s\mathcal{L}\{Y\} - Y(0)$.

<u>Proof</u>: $\mathcal{L}\{Y'\} = \int_0^\infty e^{-st} Y' \, dt$

$= \lim_{A\to\infty} \int_0^A e^{-st} Y' \, dt$ (assume $s > 0$.)

(Let $u = e^{-st}$, $dV = Y' \, dt$

$\to du = -se^{-st} dt$, $V = Y$)

$$= \lim_{A\to\infty}\left[Ye^{-st}\Big|_0^A - -s\int_0^A e^{-st}\,Y\,dt \right]$$

$$= \lim_{A\to\infty}\left[\left(\overset{0}{\cancel{\frac{Y(A)}{e^{sA}}}} - Y(0)\overset{1}{\cancel{e^{0}}}\right) + s\int_0^A e^{-st}\,Y\,dt\right]$$

since Y is of exponential order

$$= s\cdot\lim_{A\to\infty}\int_0^A e^{-st}\,Y\,dt - Y(0)$$

$$= s\cdot\int_0^\infty e^{-st}\,Y\,dt - Y(0)$$

$$= s\,\mathcal{L}\{Y\} - Y(0).$$

<u>Rule 2</u>: $\mathcal{L}\{Y''\} = s^2\mathcal{L}\{Y\} - sY(0) - Y'(0)$.

<u>Proof</u>: $\mathcal{L}\{Y''\} = \mathcal{L}\{(Y')'\}$

$$= s\,\mathcal{L}\{Y'\} - Y'(0) \quad \text{(By Rule 1)}$$

$$= s\left(s\mathcal{L}\{Y\} - Y(0)\right) - Y'(0) \quad \text{(By Rule 1)}$$

$$= s^2\mathcal{L}\{Y\} - sY(0) - Y'(0).$$

<u>Ex</u>: Use Laplace transforms to solve the following differential equations.

1.) $Y' = 2 + 3t^2$ and $Y(0) = 3 \rightarrow$

$\mathcal{L}\{Y'\} = \mathcal{L}\{2 + 3t^2\} = \mathcal{L}\{2 \cdot 1\} + \mathcal{L}\{3 \cdot t^2\} \rightarrow$

$s\mathcal{L}\{Y\} - \underset{3}{Y(0)} = 2 \cdot \frac{1}{s} + 3 \cdot \frac{2}{s^3} \rightarrow$

$\mathcal{L}\{Y\} = \frac{3}{s} + 2 \cdot \frac{1}{s^2} + \frac{3!}{s^4} \rightarrow$

$Y = 3 + 2t + t^3$

2.) $Y' = t^2 e^{-t}$ and $Y(0) = 0 \rightarrow$

$\mathcal{L}\{Y'\} = \mathcal{L}\{t^2 e^{-t}\} \rightarrow$

$s\mathcal{L}\{Y\} - \underset{0}{Y(0)} = \frac{2}{(s+1)^3} \rightarrow$

$\mathcal{L}\{Y\} = \frac{2}{s(s+1)^3} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3} \rightarrow$

$A(s+1)^3 + Bs(s+1)^2 + Cs(s+1) + Ds = 2$:

Let $s = 0$: $\boxed{A = 2}$, Let $s = -1$: $-D = 2 \rightarrow \boxed{D = -2}$;

Let $s = 1$: $2(8) + 4B + 2C - 2 = 2 \rightarrow$

$4B + 2C = -12 \rightarrow \boxed{2B + C = -6}$;

Let $s = -2$: $-2 - 2B + 2C + 4 = 2 \rightarrow \boxed{B = C} \rightarrow$

$2C + C = -6 \rightarrow \boxed{C = -2}$, $\boxed{B = -2}$; then

$\mathcal{L}\{Y\} = \frac{2}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2} - \frac{2}{(s+1)^3} \rightarrow$

$Y = 2 - 2e^{-t} - 2te^{-t} - t^2 e^{-t}$

3.) $Y'' = \cos t$ and $Y(0) = 1$, $Y'(0) = -1 \rightarrow$

$$\mathcal{L}\{Y''\} = \mathcal{L}\{\cos t\} \rightarrow$$

$$s^2\mathcal{L}\{Y\} - s\underbrace{Y(0)}_{1} - \underbrace{Y'(0)}_{-1} = \frac{s}{s^2+1} \rightarrow$$

$$s^2\mathcal{L}\{Y\} = s - 1 + \frac{s}{s^2+1} \rightarrow$$

$$\mathcal{L}\{Y\} = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s(s^2+1)}$$

$$\left(\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \rightarrow\right.$$

$$A(s^2+1) + (Bs+C)s = 1 \ :$$

$\underline{\text{Let } s=0}$: $A = 1$,

$\underline{\text{Let } s=i}$: $A(0) + (Bi+C)i = 1 \rightarrow$

$$(-B) + (C)i = (1) + (0)i \rightarrow$$

$$\left. C = 0 \text{ and } -B = 1 \rightarrow B = -1 \right) \rightarrow$$

$$\mathcal{L}\{Y\} = \frac{1}{s} - \frac{1}{s^2} + \left(\frac{1}{s} + \frac{-s}{s^2+1}\right) \rightarrow$$

$$\mathcal{L}\{Y\} = \frac{2}{s} - \frac{1}{s^2} - \frac{s}{s^2+1} \rightarrow$$

$$Y = 2 - t - \cos t$$

4.) $Y'' - 2Y' = 6e^{3t}$ and $Y(0) = 3$, $Y'(0) = 4 \rightarrow$

$$\mathcal{L}\{Y'' - 2Y'\} = \mathcal{L}\{6e^{3t}\} \rightarrow$$

$$\mathcal{L}\{Y''\} - 2\mathcal{L}\{Y'\} = 6\mathcal{L}\{e^{3t}\} \rightarrow$$

$$s^2\mathcal{L}\{Y\} - s\underbrace{Y(0)}_{3} - \underbrace{Y'(0)}_{4} - 2(s\mathcal{L}\{Y\} - \underbrace{Y(0)}_{3}) = 6 \cdot \frac{1}{s-3} \rightarrow$$

$$s^2\mathcal{L}\{Y\} - 3s - 4 - 2s\mathcal{L}\{Y\} + 6 = \frac{6}{s-3} \longrightarrow$$

$$(s^2 - 2s)\mathcal{L}\{Y\} = 3s - 2 + \frac{6}{s-3} \longrightarrow$$

$$\mathcal{L}\{Y\} = \frac{3s-2}{s(s-2)} + \frac{6}{s(s-2)(s-3)} \longrightarrow$$

$$\mathcal{L}\{Y\} = \frac{(3s-2)(s-3)+6}{s(s-2)(s-3)} \longrightarrow$$

$$\mathcal{L}\{Y\} = \frac{3s^2 - 11s + 12}{s(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3} \rightarrow$$

$$A(s-2)(s-3) + Bs(s-3) + Cs(s-2)$$
$$= 3s^2 - 11s + 12 :$$

<u>Let $s=0$</u>: $6A = 12 \rightarrow A = 2$,

<u>Let $s=2$</u>: $-2B = 2 \rightarrow B = -1$,

<u>Let $s=3$</u>: $3C = 6 \rightarrow C = 2$; then

$$\mathcal{L}\{Y\} = \frac{2}{s} + \frac{-1}{s-2} + \frac{2}{s-3} \longrightarrow$$

$$Y = 2 - e^{2t} + 2e^{3t}$$